Math 173 - Test 1

February 12, 2015

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Find a vector of magnitude 7 that has the direction from P $(2, 5, 8)$ to Q $(-1, 3, 4)$.

$$\overrightarrow{PQ} = (-1-2)\hat{\imath} + (3-5)\hat{\jmath} + (4-8)\hat{k}$$
$$= -3\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$$
$$\|\overrightarrow{PQ}\| = \sqrt{9+4+16} = \sqrt{29}$$

$$\boxed{\frac{7}{\sqrt{29}}(-3\hat{\imath} - 2\hat{\jmath} - 4\hat{k})}$$

2. (6 points) Let $\vec{u} = -5\hat{\imath} + 4\hat{\jmath} - \hat{k}$.

   (a) Find a vector, different from $\vec{u}$, that is parallel to $\vec{u}$. Give a one-sentence explanation of how you know.

   ANY SCALAR MULTIPLE OF $\vec{u}$ IS PARALLEL TO $\vec{u}$:

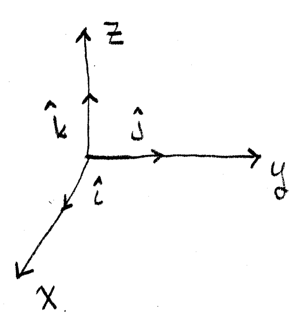
   $$2\vec{u} = \boxed{-10\hat{\imath} + 8\hat{\jmath} - 2\hat{k}}$$

   (b) Find a nonzero vector that is orthogonal to $\vec{u}$. Give a one-sentence explanation of how you know.

   IF $\vec{v} \cdot \vec{u} = 0$, THEN $\vec{v}$ IS ORTHOGONAL TO $\vec{u}$.

   $$\boxed{\vec{v} = 2\hat{\imath} + 2\hat{\jmath} - 2\hat{k}} \qquad \vec{v} \cdot \vec{u} = -10 + 8 + 2 = 0$$

3. (2 points) Without actually computing the cross product, use the right-hand rule to determine $\hat{k} \times \hat{\imath}$.



$$\boxed{\hat{k} \times \hat{\imath} = \hat{\jmath}}$$

4. (6 points) Find a set of symmetric equations for the line passing through $(2, 1, 5)$ and parallel to the line with parametric equations $\underbrace{x = 3 + 2t,\ y = -6 - t,\ z = 7 - 5t}_{\vec{u} = 2\hat{i} - \hat{j} - 5\hat{k}}$.

$$\left.\begin{array}{l}\vec{u} = 2\hat{i} - \hat{j} - 5\hat{k} \\ P(2, 1, 5)\end{array}\right\} \quad \boxed{\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-5}{-5}}$$

5. (9 points) In the following problems, the force vectors $\vec{F_1}$ and $\vec{F_2}$ are 2D vectors in the xy-plane.

(a) The force $\vec{F_1}$ has magnitude 3 and makes a 225° angle with the positive x-axis. Find the component form of $\vec{F_1}$.

$$\vec{F}_1 = 3\cos 225°\,\hat{i} + 3\sin 225°\,\hat{j}$$
$$= \boxed{-\frac{3\sqrt{2}}{2}\hat{i} - \frac{3\sqrt{2}}{2}\hat{j}}$$

(b) The force $\vec{F_2}$ has component form $\vec{F_2} = 5\sqrt{3}\hat{\imath} + 10\hat{\jmath}$. Compute the resultant vector $\vec{F} = \vec{F_1} + \vec{F_2}$.

$$\vec{F} = -\frac{3\sqrt{2}}{2}\hat{i} - \frac{3\sqrt{2}}{2}\hat{j} + 5\sqrt{3}\,\hat{i} + 10\hat{j}$$
$$\boxed{\vec{F} = \left(5\sqrt{3} - \frac{3\sqrt{2}}{2}\right)\hat{i} + \left(10 - \frac{3\sqrt{2}}{2}\right)\hat{j} \approx 6.54\hat{i} + 7.88\hat{j}}$$

(c) What angle does $\vec{F}$ make with the positive x-axis?

$\vec{F}$ IS IN THE 1^{ST} QUAD WITH

$$\tan\theta = \frac{10 - \frac{3\sqrt{2}}{2}}{5\sqrt{3} - \frac{3\sqrt{2}}{2}} \approx 1.205 \Rightarrow \boxed{\begin{array}{l}\theta \approx 0.878 \text{ RADIANS} \\ \approx 50.3°\end{array}}$$

6. (4 points) The angle between $\vec{u}$ and $\vec{v}$ is obtuse. What can be said about $\vec{u} \cdot \vec{v}$? Briefly explain.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta < 0$$

IF θ IS OBTUSE.

$$\boxed{\vec{u} \cdot \vec{v} < 0}$$

7. (6 points) Using the parameter $x = t$, find a vector-valued function (i.e., a set of parametric equations) that describes the curve of intersection of the surfaces $z = x^2 + y^2$ and $x + y = 0$.

$$x = t \Rightarrow \underbrace{y = -x}_{} \quad y = -t \implies z = x^2 + y^2 = t^2 + t^2 = 2t^2$$

$$\boxed{\vec{r}(t) = t\hat{i} - t\hat{j} + 2t^2\hat{k}}$$

8. (6 points) Find the distance from the point $P\ (7, 1, -3)$ to the plane $4x - 2y + z = 5$.

$$\text{DISTANCE} = \frac{|4(7) - 2(1) + 1(-3) - 5|}{\sqrt{16 + 4 + 1}}$$

$$= \frac{|28 - 2 - 3 - 5|}{\sqrt{21}} = \boxed{\frac{18}{\sqrt{21}}}$$

9. (6 points) Find the projection of $\vec{w} = \hat{\imath} + 4\hat{\jmath} - 3\hat{k}$ onto $\vec{u} = 7\hat{\imath} + 4\hat{k}$.

$$\text{proj}_{\vec{u}}\,\vec{w} = \frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}}\,\vec{u} = \frac{7 + 0 - 12}{49 + 16}\,\vec{u}$$

$$= \frac{-5}{65}\,\vec{u} = \boxed{-\frac{5}{65}(7\hat{i} + 4\hat{k})}$$

10. (8 points) Find a set of parametric equations for a line in the plane $5x - 9y - 8z = 5$.

TWO POINTS IN THE PLANE: $(1,0,0)$ & $(3,2,-1)$

$P \qquad Q$

$\vec{PQ} = 2\hat{i} + 2\hat{j} - \hat{k}$

POINT $(1,0,0)$

$\Downarrow$

$$\boxed{x = 1 + 2t, \quad y = 2t, \quad z = -t}$$

11. (7 points) Find the vector-valued function $\vec{r}(t)$ such that

$$\vec{r}\,'(t) = te^{-t^2}\hat{\imath} - e^{-t}\hat{\jmath} + \hat{k}; \quad \vec{r}(0) = \frac{1}{2}\hat{\imath} - \hat{\jmath} + 2\hat{k}.$$

$$\vec{r}(t) = \left(-\tfrac{1}{2}e^{-t^2} + c_1\right)\hat{i} + \left(e^{-t} + c_2\right)\hat{j} + (t + c_3)\hat{k}$$

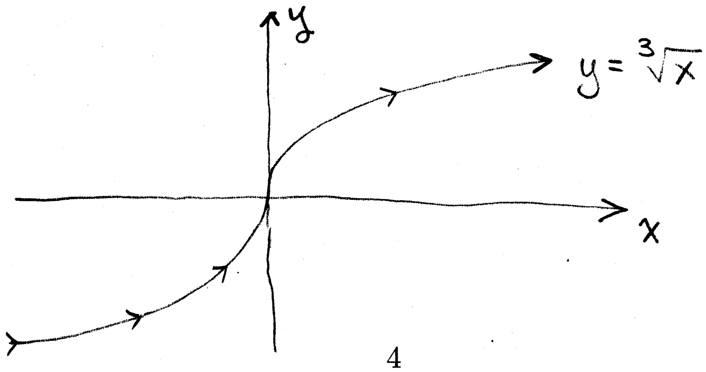
$$\vec{r}(0) = \left(-\tfrac{1}{2} + c_1\right)\hat{i} + (1 + c_2)\hat{j} + c_3\hat{k} = \tfrac{1}{2}\hat{i} - \hat{j} + 2\hat{k}$$

$$\Rightarrow \quad c_1 = 1, \quad c_2 = -2, \quad c_3 = 2$$

$$\boxed{\vec{r}(t) = \left(1 - \tfrac{1}{2}e^{-t^2}\right)\hat{i} + \left(e^{-t} - 2\right)\hat{j} + (t + 2)\hat{k}}$$

12. (6 points) Sketch the graph of the vector-valued function $\vec{r}(t) = t^3\hat{\imath} + t\hat{\jmath}$. Draw arrows on your graph to indicate the curve's orientation.

$x = t^3$
$y = t$ $\quad \Rightarrow \quad x = y^3 \quad$ OR $\quad y = \sqrt[3]{x}$



13. (10 points) Let $\vec{r}(t) = 2\cos t\,\hat{\imath} - e^t\hat{\jmath} + t^2\hat{k}$.

(a) Let $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$. Compute $\vec{T}(t)$.

$$\vec{r}\,'(t) = -2\sin t\,\hat{\imath} - e^t\,\hat{\jmath} + 2t\,\hat{k} \qquad \|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + e^{2t} + 4t^2}$$

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{4\sin^2 t + e^{2t} + 4t^2}}\left(-2\sin t\,\hat{\imath} - e^t\hat{\jmath} + 2t\hat{k}\right)}$$

(b) If you were forced to compute $\vec{T}(t) \cdot \vec{T}\,'(t)$, what would you find? Explain.

$\vec{T} \cdot \vec{T}\,' = 0$ BECAUSE $\vec{T}(t)$ IS A UNIT VECTOR,

IT HAS CONSTANT MAGNITUDE,

$\therefore$ IT IS ORTHOGONAL TO ITS DERIVATIVE.

14. (10 points) Let $\vec{u} = -2\hat{\imath} + 9\hat{\jmath} + \hat{k}$ and $\vec{v} = \hat{\imath} - \hat{\jmath} + 4\hat{k}$.

(a) Find a vector orthogonal to both $\vec{u}$ and $\vec{v}$. $\vec{u} \times \vec{v}$ IS ORTHOG TO BOTH.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 9 & 1 \\ 1 & -1 & 4 \end{vmatrix} = \hat{\imath}\,(36+1) - \hat{\jmath}\,(-8-1) + \hat{k}\,(2-9)$$

$$= \boxed{37\hat{\imath} + 9\hat{\jmath} - 7\hat{k}}$$

(b) Find an equation of the plane passing through $(5, 0, 3)$ with normal vector is $\vec{u} \times \vec{v}$.

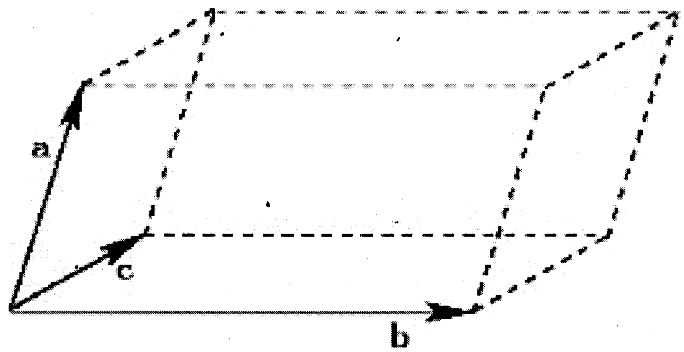
$\vec{N} = 37\hat{\imath} + 9\hat{\jmath} - 7\hat{k}$

POINT $(5, 0, 3)$

$$37(x-5) + 9(y-0) - 7(z-3) = 0$$

$$\boxed{37x + 9y - 7z = 164}$$

15. (8 points) A crystal structure has the form of a parallelepiped determined by the vectors $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = 3\hat{\jmath} + 5\hat{k}$, and $\vec{c} = -4\hat{\imath} + 2\hat{\jmath} + \hat{k}$, where distances are measured in micrometers. Find the volume of the parallelepiped.



$$\text{Volume} = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ -4 & 2 & 1 \end{vmatrix} = 1(3-10) - 2(0+20) + 1(0+12)$$

$$= -7 - 40 + 12 = -35$$

$$\boxed{\text{Volume} = 35\ \mu m^3}$$