Math 173 - Test 2

March 26, 2015

Name key

Score

Show all work. Supply explanations when necessary.

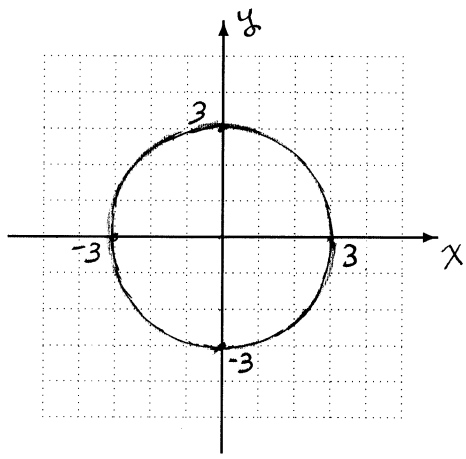
1. (8 points) Suppose the equation $xyz + 3xy^2 + xe^{yz} - 4x = 0$ implicitly defines z as a function of x and y. Find $\partial z/\partial y$ at the point $(1, 1, 0)$.

LET $F(x,y,z) = xyz + 3xy^2 + xe^{yz} - 4x$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(xz + 6xy + xze^{yz})}{xy + xye^{yz}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1,0)} = \frac{-(0+6+0)}{1+1} = \boxed{-3}$$

2. (4 points) Sketch a curve whose curvature function is the constant function $\kappa = 1/3$. Briefly explain.



A CIRCLE OF RADIUS 3 HAS CONSTANT CURVATURE $\frac{1}{3}$.

3. (10 points) Consider the function $P(x, y) = \ln(4 + x^2 + y^2)$.

(a) Find the directional derivative of P at the point $(-1, 2)$ in the direction of $\vec{w} = 2\hat{\imath} + \hat{\jmath}$.

$$\vec{\nabla} P(x,y) = \frac{2x}{4+x^2+y^2}\hat{\imath} + \frac{2y}{4+x^2+y^2}\hat{\jmath}, \quad \vec{\nabla} P(-1,2) = -\frac{2}{9}\hat{\imath} + \frac{4}{9}\hat{\jmath}$$

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{2}{\sqrt{5}}\hat{\imath} + \frac{1}{\sqrt{5}}\hat{\jmath}$$

$$D_{\vec{w}} P(-1,2) = \left(-\frac{2}{9}\right)\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{4}{9}\right)\left(\frac{1}{\sqrt{5}}\right)$$

$$= \boxed{0}$$

(b) At the point $(-1, 2)$, what is the direction of steepest descent (maximum decrease) of P?

$$-\vec{\nabla} P(-1,2) = \frac{2}{9}\hat{\imath} - \frac{4}{9}\hat{\jmath}$$

STEEPEST DESCENT IN DIRECTION OF $\hat{\imath} - 2\hat{\jmath}$

4. (10 points) The *body mass index* (BMI) for an adult human is given by $B = 703w/h^2$, where w is weight in pounds and h is height in inches. Suppose you weigh 190 lbs and your height is 70 in. Your weight and height measurements have possible errors $\Delta w = \pm 1.5$ lbs and $\Delta h = \pm 0.5$ in. Use differentials to estimate the error in your BMI.

$$B = \frac{703w}{h^2} \qquad dB = \frac{703}{h^2}dw - \frac{1406w}{h^3}dh$$

$$\Delta B \approx \frac{703}{h^2}\Delta w - \frac{1406w}{h^3}\Delta h$$

$$\Delta B \approx \frac{703}{70^2}(1.5) - \frac{1406(190)}{70^3}(0.5)$$

$$\approx -0.1742$$

$\Rightarrow$ ERROR IN BMI IS ± 0.1742

5. (6 points) Suppose z is a function of x, y; and x, y are functions of t, u, v. Write the chain rule formulas for $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

6. (10 points) Evaluate each limit or show that it does not exist.

(a) $\lim_{(x,y)\to(1,1)} \frac{x^2 + xy - 2y^2}{2x^2 - xy - y^2}$ $\quad \frac{0}{0}$

$$= \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(x+2y)}{\cancel{(x-y)}(2x+y)} = \frac{3}{3} = \boxed{1}$$

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$ $\quad \frac{0}{0}$

Along $x = 0$: $\lim_{y\to 0} \frac{0}{y^2} = 0$

Along $y = 0$: $\lim_{x\to 0} \frac{x^2}{x^2} = 1$

$\boxed{\text{LIMIT DNE}}$

7. (12 points) At time $t = 0$, a baseball is hit 3 ft above the ground at an angle of 45° with a speed of $80\sqrt{2}$ ft/s. Neglect all forces other than gravity. (Use $g = 32\,\text{ft/s}^2$.)

(a) Find the vector-valued functions that give the position and velocity of the ball at time t.

$$\vec{r}(t) = 80\sqrt{2}\cos 45^\circ\, t\,\hat{\imath} + \left(-\tfrac{1}{2}gt^2 + 80\sqrt{2}\sin 45^\circ\, t + 3\right)\hat{\jmath}$$

$$\vec{r}(t) = 80t\,\hat{\imath} + \left(-16t^2 + 80t + 3\right)\hat{\jmath}$$

$$\vec{v}(t) = 80\hat{\imath} + \left(-32t + 80\right)\hat{\jmath}$$

(b) What is the maximum height of the ball?

$$-32t + 80 = 0 \Rightarrow t = 80/32 = 2.5\text{ s}$$

$$-16(2.5)^2 + 80(2.5) + 3 = \boxed{103\text{ FT}}$$

(c) Will the ball clear a 20-ft fence that is 380 ft downrange?

$$80t = 380 \Rightarrow t = 380/80 = 4.75\text{ s}$$

$$-16(4.75)^2 + 80(4.75) + 3 = 22\text{ FT}$$

YES, IT WILL CLEAR BY 2 FT.

8. (12 points) Let $\vec{r}(t) = \frac{t^2}{2}\hat{\imath} + (4 - 3t)\hat{\jmath} + 2\hat{k}$. Find the principal unit normal vector at the point where $t = 0$.

$$\vec{r}\,'(t) = t\hat{\imath} - 3\hat{\jmath}$$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2+9}$$

$$\hat{T}(t) = \frac{t\hat{\imath} - 3\hat{\jmath}}{\sqrt{t^2+9}}$$

$$\hat{T}'(t) = \frac{\sqrt{t^2+9}\,(\hat{\imath}) - (t\hat{\imath} - 3\hat{\jmath})\frac{1}{2}(t^2+9)^{-1/2}(2t)}{t^2+9}$$

$$\hat{T}'(0) = \frac{3\hat{\imath}}{9} = \frac{1}{3}\hat{\imath}$$

$$\hat{N}(0) = \frac{\hat{T}'(0)}{\|\hat{T}'(0)\|} = \frac{\frac{1}{3}\hat{\imath}}{\frac{1}{3}} = \boxed{\hat{\imath}}$$

9. (10 points) Consider the function $g(x, y) = \ln(x^2 + y)$.

$x^2 + y > 0 \Rightarrow$

$y > -x^2$

(a) What is the domain of g?

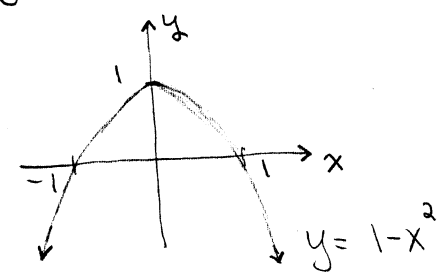
$\{(x,y) : y > -x^2\}$

(b) Discuss the continuity of g.

g IS CONTINUOUS EVERYWHERE IT IS DEFINED.

(c) Sketch the level curve $g(x, y) = 0$.

$\ln(x^2+y) = 0 \Rightarrow x^2+y = 1 \Rightarrow y = 1-x^2$



(d) Compute the mixed partial derivative g_{xy}.

$$g_x(x,y) = \frac{2x}{x^2+y}$$

$$g_{xy}(x,y) = \frac{-2x}{(x^2+y)^2}$$ BY QUOTIENT RULE

(e) Without actually computing the mixed partial derivative g_{yx}, would you expect it to be equal to g_{xy}? Explain.

YES, THE MIXED PARTIAL DERIVATIVES WILL BE CONTINUOUS WHEREVER THEY ARE DEFINED. BY THE THEOREM FROM CLASS, THEY WILL BE EQUAL.

10. (8 points) For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, let $f(x) = \ln(\cos x)$. Find the curvature function $\kappa(x)$.

$$K(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$$

$$f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

$$f''(x) = -\sec^2 x$$

$$K(x) = \frac{\sec^2 x}{(1+\tan^2 x)^{3/2}} = \frac{\sec^2 x}{\sec^3 x} = \frac{1}{\sec x}$$

$$= \cos x$$

$$\boxed{K(x) = \cos x}$$

11. (4 points) Consider the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$. Do you expect that f is a differentiable function? Where? How do you know?

According to a theorem from class, f will be differentiable everywhere that f_x and f_y are continuous. f_x & f_y will be continuous for all $(x,y) \neq (0,0)$.

12. (6 points) Let $g(x, y, z) = 2x^2yz - \cos(xy) + z^3e^{-x}$. Find $\vec{\nabla} g(x, y, z)$.

$$\vec{\nabla} g(x,y,z) = \left(4xyz + y\sin(xy) - z^3e^{-x}\right)\hat{i}$$
$$+ \left(2x^2z + x\sin(xy)\right)\hat{j}$$
$$+ \left(2x^2y + 3z^2e^{-x}\right)\hat{k}$$