Math 173 - Test 1

February 15, 2018

Name key

Score

Show all work. Supply explanations when necessary.

1. (4 points) Let $\vec{w} = -3\hat{\imath} - \hat{\jmath} + 4\hat{k}$. Find a vector of magnitude 6 in the direction opposite of $\vec{w}$.

$-\vec{w}$

$$\|\vec{\omega}\| = \sqrt{9+1+16} = \sqrt{26}$$

$$-\frac{6}{\sqrt{26}}\vec{\omega} = \frac{6}{\sqrt{26}}\left(3\hat{\imath} + \hat{\jmath} - 4\hat{k}\right)$$

2. (8 points) Let P and Q be the points $(3, 5, -2)$ and $(5, 3, -6)$, respectively.

(a) Find the midpoint of the line segment connecting P and Q.

$$R\left(\frac{3+5}{2}, \frac{5+3}{2}, \frac{-2+-6}{2}\right)$$

$$R\,(4, 4, -4)$$

(b) Find a nonzero vector that is orthogonal to $\vec{PQ}$.

LOTS OF POSSIBLE ANSWERS

$$\vec{PQ} = 2\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$$

$$\vec{u} = \hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\vec{u} \cdot \vec{PQ} = 2 + 2 - 4 = 0$$

(c) Find a set of parametric equations for the line passing through the midpoint of segment PQ and parallel to the vector you found in part (b).

LOTS OF POSSIBLE ANSWERS

$\vec{u} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$R\,(4, 4, -4)$

$$x = 4 + t$$
$$y = 4 - t$$
$$z = -4 + t$$

3. (6 points) Forces with magnitudes of 500 lb and 200 lb act on a machine part at angles of 30° and −45°, respectively, with the positive x-axis. Find the magnitude of the resultant force.

$$\vec{F}_1 = 500\cos 30^\circ\,\hat{i} + 500\sin 30^\circ\,\hat{j} = 250\sqrt{3}\,\hat{i} + 250\,\hat{j}$$

$$\vec{F}_2 = 200\cos(-45^\circ)\,\hat{i} + 200\sin(-45^\circ)\,\hat{j} = 100\sqrt{2}\,\hat{i} - 100\sqrt{2}\,\hat{j}$$

$$\|\vec{F}_1 + \vec{F}_2\| = \sqrt{(250\sqrt{3} + 100\sqrt{2})^2 + (250 - 100\sqrt{2})^2}$$

$$\approx 584.6 \text{ lbs}$$

4. (12 points) Consider the triangle with vertices $A(5,3,1)$, $B(3,2,3)$, and $C(-4,-1,2)$.

(a) Compute $\vec{AB} \times \vec{AC}$.

$$\vec{AB} = -2\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{AC} = -9\hat{i} - 4\hat{j} + \hat{k}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 2 \\ -9 & -4 & 1 \end{vmatrix}$$

$$= 7\hat{i} - 16\hat{j} - \hat{k}$$

(b) Find the area of $\triangle ABC$.

$$\frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{49 + 256 + 1} = \frac{\sqrt{306}}{2} \approx 8.75$$

(c) Find an equation of the plane containing $\triangle ABC$.

$$\vec{n} = \vec{AB} \times \vec{AC} = 7\hat{i} - 16\hat{j} - \hat{k}$$

Point $A(5,3,1)$

$$7(x-5) - 16(y-3) - (z-1) = 0$$

$$7x - 16y - z = -14$$

5. (8 points) Consider the planes described by the following equations.

$$2x - 3y - z = -5 \quad \text{and} \quad 6x - 9y - 3z = 18$$

(a) Are the planes parallel? Explain.

$\vec{n}_1 = 2\hat{i} - 3\hat{j} - \hat{k}$

$\vec{n}_2 = 6\hat{i} - 9\hat{j} - 3\hat{k}$

PLANES ARE PARALLEL SINCE $3\vec{n}_1 = \vec{n}_2$.

(b) Find the distance between the planes.

PLANE 1: $2x - 3y - z = -5$ PLANE 2: $2x - 3y - z = 6$

SAY (x_0, y_0, z_0) IS ON THIS PLANE

$$\text{DISTANCE} = \frac{|2x_0 - 3y_0 - z_0 - 6|}{\sqrt{4+9+1}} = \frac{|-5-6|}{\sqrt{14}} = \boxed{\frac{11}{\sqrt{14}}} \approx 2.94$$

6. (6 points) Find the angle between the vectors $\vec{v} = -2\hat{\imath} + \hat{\jmath} + 6\hat{k}$ and $\vec{w} = 4\hat{\imath} + 3\hat{k}$.

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{-8+0+18}{\sqrt{4+1+36}\sqrt{16+9}} = \frac{10}{5\sqrt{41}} = \frac{2}{\sqrt{41}} \Rightarrow \boxed{\theta \approx 71.8^\circ}$$

7. (4 points) Find the domain of the vector-valued function
$\vec{f}(t) = \sqrt{t+2}\,\hat{\imath} + (5\cos t)\,\hat{\jmath} + \dfrac{1}{\sin t}\,\hat{k}$.

$t \geq -2$ $t \in \mathbb{R}$ $\sin t \neq 0$

$\{t : t \geq -2$ AND t IS NOT A MULTIPLE OF $\pi\}$

8. (6 points) Suppose $\frac{d\vec{r}}{dt} = (\sec^2 t)\hat{\imath} + (\sin t)\hat{\jmath} + 5e^{-2t}\hat{k}$. Find $\vec{r}(t)$ if $\vec{r}(0) = 4\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$.

$$\vec{r}(t) = (\tan t + c_1)\hat{\imath} + (-\cos t + c_2)\hat{\jmath} + \left(-\frac{5}{2}e^{-2t} + c_3\right)\hat{k}$$

$$\vec{r}(0) = 4\hat{\imath} + 2\hat{\jmath} - 8\hat{k} = c_1\hat{\imath} + (c_2 - 1)\hat{\jmath} + \left(c_3 - \frac{5}{2}\right)\hat{k}$$

$$\Rightarrow c_1 = 4, \; c_2 = 3, \; c_3 = -\frac{11}{2}$$

$$\vec{r}(t) = (4 + \tan t)\hat{\imath} + (3 - \cos t)\hat{\jmath} + \left(-\frac{11}{2} - \frac{5}{2}e^{-2t}\right)\hat{k}$$

9. (10 points) Consider the surface described by the equation $2z^2 = x^2 + y^2 - 4$.

(a) Identify the surface.

$$x^2 + y^2 = 2z^2 + 4$$

HYPERBOLOID OF ONE SHEET

(b) Describe (or sketch) in detail the level curve obtained by fixing $z = 0$.

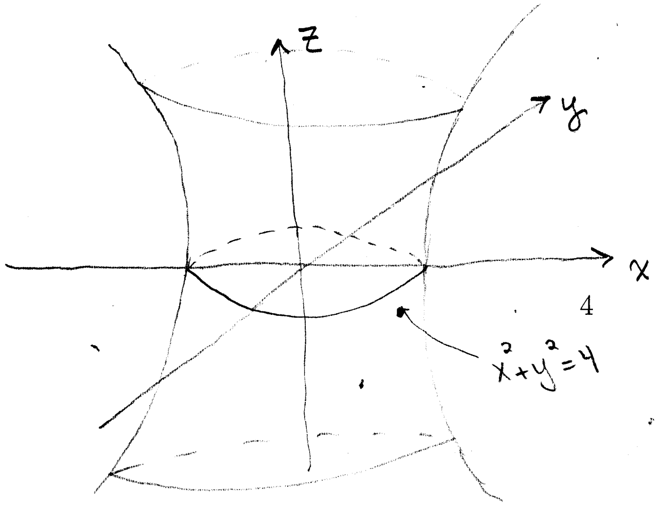
$x^2 + y^2 = 4 \longrightarrow$ CIRCLE CENTERED AT $(0,0)$ IN XY-PLANE, WITH RADIUS 2.

(c) Describe (or sketch) in detail the level curve(s) obtained by fixing $y = 2$.

$$2z^2 = x^2 \Rightarrow z = \frac{1}{\sqrt{2}}x, \quad z = -\frac{1}{\sqrt{2}}x$$

LINE THRU ORIGIN

(d) Sketch a rough graph of the surface.



10. (12 points) A projectile is launched from a bench above the ground. It is launched with an initial speed of 100 feet per second and at an angle of 30° above the horizontal. The projectile hits the ground after it has covered a horizontal distance of 276 ft. How high is the bench?

$$\vec{r}(t) = (100\cos 30^\circ t)\,\hat{i} + (-16t^2 + 100\sin 30^\circ t + y_0)\,\hat{j}$$

$$= 50\sqrt{3}\,t\,\hat{i} + (-16t^2 + 50t + y_0)\,\hat{j}$$

$-16t^2 + 50t + y_0 = 0$ WHEN $\underbrace{50\sqrt{3}\,t = 276}_{t = \frac{276}{50\sqrt{3}}}$

$$y_0 = 16t^2 - 50t$$

$$= 16\left(\frac{276}{50\sqrt{3}}\right)^2 - 50\left(\frac{276}{50\sqrt{3}}\right) \approx \boxed{3.16 \text{ FT}}$$

11. (8 points) A particle is moving along the helix described by the vector-valued function $\vec{r}(t) = (3\sin t)\hat{\imath} + (3\cos t)\hat{\jmath} + 5t\hat{k}$. Find the particle's unit tangent vector.

$$\vec{r}\,'(t) = 3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 5\hat{k}$$

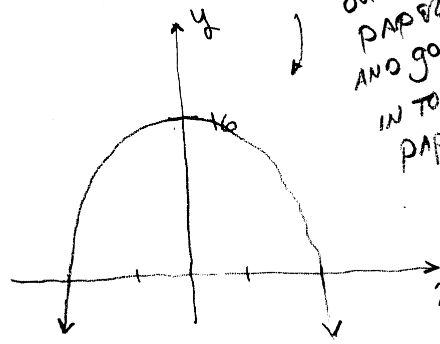
$$\|\vec{r}\,'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 25} = \sqrt{34}$$

$$\boxed{\hat{T}(t) = \frac{1}{\sqrt{34}}(3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 5\hat{k})}$$

12. (4 points) Sketch or describe the <u>surface in space</u> defined by the equation $y = 16 - x^4$.

$y = 16 - x^4$

DESCRIBES THE CYLINDER IN SPACE CONSISTING OF ALL LINES PARALLEL TO THE Z-AXIS AND PASSING THROUGH THE GENERATING CURVE $y = 16 - x^4$



THIS COMING OUT OF PAPER AND GOING IN TO PAPER.

13. (4 points) Determine a vector-valued function whose graph is the plane curve described by the equation $xy + x^2 = 3$. $\longrightarrow$ $y = \dfrac{3 - x^2}{x}$

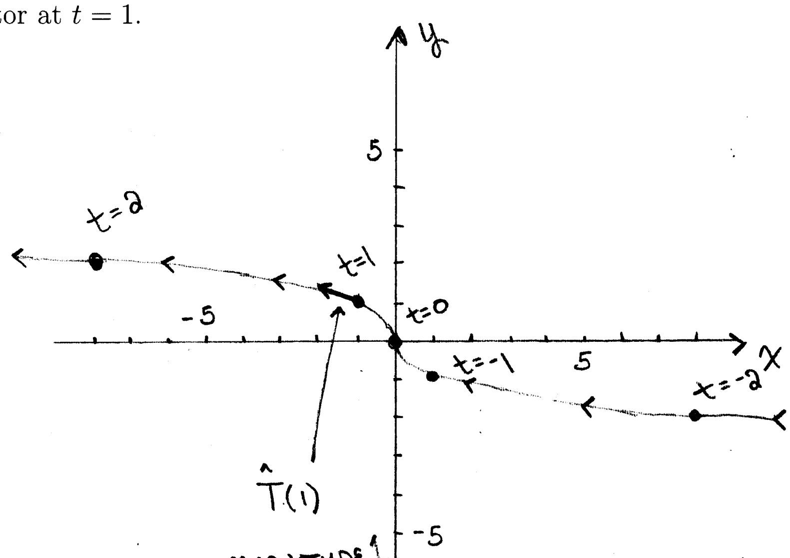
$x = t$

$y = \dfrac{3 - t^2}{t}$, $t \neq 0$

$$\vec{r}(t) = t\,\hat{\imath} + \frac{3 - t^2}{t}\,\hat{\jmath}$$

14. (8 points) Sketch the graph of the vector-valued function $\vec{r}(t) = -t^3\hat{\imath} + t\hat{\jmath}$. Draw arrows on your graph to indicate the curve's orientation. Then, without actually computing it, sketch the unit tangent vector at $t = 1$.

t	$(-t^3, t)$
0	(0,0)
1	(-1,1)
-1	(1,-1)
2	(-8,2)
-2	(8,-2)



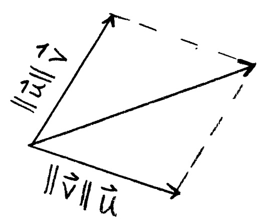
MAGNITUDE 1, TANGENT TO CURVE, POINTS IN DIRECTION OF MOTION

15. (4 points extra credit) For nonzero vectors $\vec{u}$ and $\vec{v}$, prove that the vector $\vec{w} = \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}$ bisects the angle between $\vec{u}$ and $\vec{v}$. (You may assume basic geometry facts about parallelograms.)

$\|\vec{u}\|\vec{v}$ & $\|\vec{v}\|\vec{u}$

HAVE THE SAME MAGNITUDE, NAMELY $\|\vec{u}\|\,\|\vec{v}\|$.

SO, $\|\vec{u}\|\vec{v}$ AND $\|\vec{v}\|\vec{u}$ FORM THE SIDES OF A RHOMBUS



$\vec{w}$ IS THE DIAGONAL OF A RHOMBUS.

THE DIAGONALS OF A RHOMBUS BISECT OPPOSITE ANGLES.