Math 173 - Test 2
March 21, 2018

Name <u>Key</u>

Score ______

Show all work. Supply explanations when necessary.

1. (10 points) Let $g(x, y) = \ln(x^2 - y)$.

   (a) What is the domain of g?

   $x^2 - y > 0$

   $\Rightarrow y < x^2$

   $$\boxed{\text{Domain} = \{(x,y) \in \mathbb{R}^2 : y < x^2\}}$$

   (b) Evaluate $g(e, 0)$.

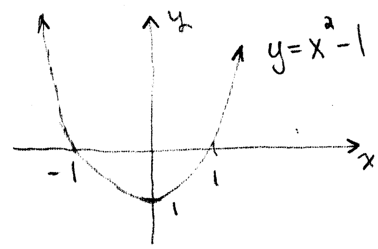
   $$g(e,0) = \ln(e^2 - 0) = \ln e^2 = \boxed{2}$$

   (c) Sketch the level curve $g(x, y) = 0$.

   $\ln(x^2 - y) = 0$

   $\Rightarrow x^2 - y = 1$

   $\Rightarrow y = x^2 - 1$

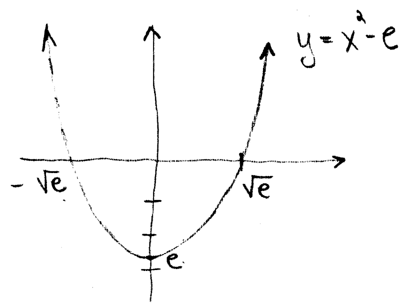


   (d) Sketch the level curve $g(x, y) = 1$.

   $\ln(x^2 - y) = 1$

   $\Rightarrow x^2 - y = e$

   $\Rightarrow y = x^2 - e$



   (e) $z = g(x, y)$ explicitly defines z as a function of x and y and implicitly defines y as a function of x and z. Find a formula that **explicitly** defines y as a function of x and z.

   $z = \ln(x^2 - y)$

   $\Rightarrow e^z = x^2 - y \Rightarrow$

   $$\boxed{y = x^2 - e^z}$$

2. (10 points) Consider the function f defined by

$$f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Determine $\lim_{(x,y)\to(0,0)} f(x,y)$ or show that it does not exist.

ALONG $y=x$:

$$\lim_{x\to 0} \frac{3x^3}{x^2+x^4} = \lim_{x\to 0} \frac{3x}{1+x^2} = 0$$

ALONG $x=y^2$:

$$\lim_{y\to 0} \frac{3y^4}{2y^4} = \frac{3}{2}$$

$\boxed{\text{LIMIT DNE}}$

(b) Discuss the continuity of f.

BEING A RATIONAL FUNCTION, $\frac{3xy^2}{x^2+y^4}$ IS CONT. EVERYWHERE IT IS DEFINED, I.E. EVERYWHERE EXCEPT $(0,0)$.

SINCE $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$, f IS CONT. EVERYWHERE EXCEPT $(0,0)$.

3. (10 points) Let $h(x,y) = e^{x^2+xy+y^2}$.

(a) Find all values of x and y for which $h_x(x,y) = 0$ and $h_y(x,y) = 0$ simultaneously.

$$f_x(x,y) = (2x+y)e^{x^2+xy+y^2} = 0 \Rightarrow 2x+y=0 \Rightarrow y=-2x$$

$$f_y(x,y) = (2y+x)e^{x^2+xy+y^2} = 0 \Rightarrow 2y+x=0 \quad -4x+x=0$$

$$\Rightarrow x=0$$

$$y=0$$

(b) If you were asked to compute $h_{xy}(x,y)$ and $h_{yx}(x,y)$, would you expect them to be equal? Briefly explain.

YES, BECAUSE OF THE KIND OF FUNCTION THAT h IS, I EXPECT h_{xy} AND h_{yx} TO BE CONTINUOUS EVERYWHERE $(\mathbb{R}^2)$

BY OUR THEOREM, THE MIXED PARTIALS WILL BE EQUAL EVERYWHERE.

4. (10 points) The volume of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.

(a) Assume that r and h are functions of the variable t. Use the chain rule to write a formula for dV/dt.

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \Rightarrow \frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

(b) Now suppose that $r = e^t$ and $h = e^{-2t}$, for $t \geq 0$. Use your result from (a) to find dV/dt.

$$\frac{dV}{dt} = (2\pi r h)(e^t) + (\pi r^2)e^{-2t} = (2\pi e^t e^{-2t})(e^t) + (\pi e^{2t})(-2e^{-2t})$$

$$= 2\pi - 2\pi = 0$$

(c) As t increases, does the volume of the cylinder in part (b) increase or decrease? Briefly explain how you know.

NEITHER!

$$\frac{dV}{dt} = 0 \Rightarrow \boxed{V \text{ IS CONSTANT}}$$

5. (10 points) Let $w = x^2yz^2 + \sin(yz)$.

(a) Determine the total differential dw.

$$dw = 2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$$

(b) Suppose (x, y, z) changes from $(2, 0, 1)$ to $(1.9, -0.2, 1.2)$. Use differentials to approximate Δw.

$\Delta x = -0.1, \ \Delta y = -0.2, \ \Delta z = 0.2$

$$\Delta w \approx (2)(2)(0)(1)^2(-0.1) + ((2)^2(1)^2 + (1)\cos(0))(-0.2)$$
$$+ ((2)(2)^2(0)(1) + (0)\cos(0))(0.2)$$

$$= 5(-0.2) = \boxed{-1}$$

6. (10 points) Let $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$.

(a) What is the domain of f?

$$x^2 + y^2 + z^2 \neq 0 \Rightarrow \text{DOMAIN IS } \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \neq (0,0,0)\}$$

(b) Evaluate $f(1, 2, 3)$.

$$f(1,2,3) = \frac{1}{\sqrt{1+4+9}} = \boxed{\frac{1}{\sqrt{14}}}$$

(c) Discuss the continuity of f.

f IS CONTINUOUS EVERYWHERE IT IS DEFINED.

(d) Describe the level surface $f(x, y, z) = 1$.

$$f(x,y,z) = 1 \Rightarrow \sqrt{x^2+y^2+z^2} = 1 \Rightarrow \underbrace{x^2+y^2+z^2 = 1}$$

UNIT SPHERE CENTERED AT ORIGIN.

7. (12 points) Evaluate each limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2 + xy + y^2}$

POLAR... $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^2(\cos\theta - \sin\theta)^2}{r^2(1+\cos\theta\sin\theta)} = \frac{(\cos\theta-\sin\theta)^2}{1+\cos\theta\sin\theta}$$

LIMIT DEPENDS ON θ.

LIMIT DNE.

(b) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{y\sin x}{x(y+1)}$

$$\lim_{(x,y)\to(0,1)} \left(\frac{y}{y+1}\right)\left(\frac{\sin x}{x}\right)$$

$$= \left(\lim_{y\to 1} \frac{y}{y+1}\right)\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = \frac{1}{2}\cdot 1 = \boxed{\frac{1}{2}}$$

8. (10 points) Assume that the following equation implicitly defines z as a function of x and y. Find the first partial derivatives of z.

$$\underbrace{x \ln y + y^2 z + z^2}_{F(x,y,z)} = 8$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(\ln y)}{y^2 + 2z} \qquad \boxed{\frac{\partial z}{\partial x} = \frac{-\ln y}{y^2+2z}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\left(\frac{x}{y} + 2yz\right)}{y^2 + 2z} \qquad \boxed{\frac{\partial z}{\partial y} = \frac{-(x + 2y^2 z)}{y^3 + 2yz}}$$

9. (8 points) Consider the function $f(x, y, z) = x^2y + 2xz^2 - 3y^2z$. Choose two 3rd-order partial derivatives that you expect to be equal. Then compute both of them.

$$f_{xyz} = f_{zxy}$$

$$f_x = 2xy + 2z^2 \qquad f_z = 4xz - 3y^2$$

$$f_{xy} = 2x \qquad f_{zx} = 4z$$

$$f_{xyz} = 0 \qquad f_{zxy} = 0$$

Intentionally blank.

$g_x = 3x^2 - 2y, \quad g_y = -2x + 7$

10. (5 points) Use the definition of differentiability to show that $g(x,y) = x^3 - 2xy + 7y$ is differentiable on $\mathbb{R}^2$.

$$\Delta z = g(x+\Delta x, y+\Delta y) - g(x,y)$$

$$= \left[(x+\Delta x)^3 - 2(x+\Delta x)(y+\Delta y) + 7(y+\Delta y)\right] - \left[x^3 - 2xy + 7y\right]$$

$$= \cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - \cancel{2xy} - 2x\Delta y - 2y\Delta x - 2\Delta x\Delta y + \cancel{7y} + 7\Delta y - \cancel{x^3} + \cancel{2xy} - \cancel{7y}$$

$$= 3x^2\Delta x - 2y\Delta x + 7\Delta y - 2x\Delta y + 3x\Delta x\Delta x + \Delta x^2\Delta x - 2\Delta x\Delta y$$

$$= (3x^2 - 2y)\Delta x + (7 - 2x)\Delta y + (3x\Delta x + \Delta x^2)\Delta x + (-2\Delta x)\Delta y$$

$$= g_x(x,y)\Delta x + g_y(x,y)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

WHERE $\epsilon_1 = 3x\Delta x + \Delta x^2$

AND $\epsilon_2 = -2\Delta x$

AND $(\epsilon_1, \epsilon_2) \to (0,0)$ AS $(\Delta x, \Delta y) \to (0,0)$.

THIS HOLDS FOR ALL (x,y) IN $\mathbb{R}^2$.

g IS DIFFERENTIABLE EVERYWHERE.

11. (5 points) The period T of a pendulum of length L is given by $T = 2\pi\sqrt{L}/\sqrt{g}$, where g is the acceleration due to gravity. A pendulum is moved from the Canal Zone, where $g = 32.09\,\text{ft/s}^2$, to Greenland, where $g = 32.23\,\text{ft/s}^2$. Because of the change in temperature, the length of the pendulum shrank from 2.5 ft to 2.48 ft. Use differentials to approximate the change in the period of the pendulum.

$$\Delta T \approx \frac{\partial T}{\partial L}\Delta L + \frac{\partial T}{\partial g}\Delta g$$

$$\approx \frac{\pi}{\sqrt{Lg}}\Delta L - \frac{\pi\sqrt{L}}{g^{3/2}}\Delta g$$

$$\Delta g = 32.23 - 32.09 = 0.14$$

$$\Delta L = 2.48 - 2.5 = -0.02$$

$$\Delta T \approx \frac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) - \frac{\pi\sqrt{2.5}}{(32.09)^{3/2}}(0.14)$$

$$\approx -0.01084 \text{ SECONDS}$$