Math 173 - Test 3a

April 26, 2018

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Let $g(x, y) = \sqrt{20 + x^2 + 2xy - y^2}$.

   (a) Find a unit vector in the direction of maximum increase of g at the point $(1, 2)$.

$$\vec{\nabla} g(x,y) = \frac{1}{2}(20+x^2+2xy-y^2)^{-1/2}\left((2x+2y)\hat{\imath} + (2x-2y)\hat{\jmath}\right)$$

$$\vec{\nabla} g(1,2) = \frac{1}{2}(20+1+4-4)^{-1/2}(6\hat{\imath}-2\hat{\jmath}) = \frac{1}{\sqrt{21}}(3\hat{\imath}-\hat{\jmath})$$

$$\|\vec{\nabla} g(1,2)\| = \frac{1}{\sqrt{21}}\sqrt{9+1} = \frac{\sqrt{10}}{\sqrt{21}}$$

UNIT VECTOR IS $\frac{1}{\sqrt{10}}(3\hat{\imath}-\hat{\jmath})$

   (b) Find the directional derivative of g at the point $(1, 2)$ in the direction of the vector you found in part (a). (If you were unable to do part (a), just use any vector with nonzero components.)

DIRECTIONAL DERIVATIVE IN DIRECTION OF MAX INCREASE AT $(1,2)$

$$= \boxed{\|\vec{\nabla} g(1,2)\| = \frac{\sqrt{10}}{\sqrt{21}}}$$

2. (4 points) Suppose $z = f(x, y)$ is a differentiable function and (x_0, y_0, z_0) is a point on its graph. Is it true that $\vec{\nabla} f(x_0, y_0)$ is normal to the graph of f at (x_0, y_0, z_0)? Explain your reasoning.

No, $\vec{\nabla} f(x_0,y_0)$ IS A 2D VECTOR. IT IS NORMAL TO THE LEVEL CURVE $z_0 = f(x,y)$ AT (x_0,y_0).

LET $F(x,y,z) = z - f(x,y)$.

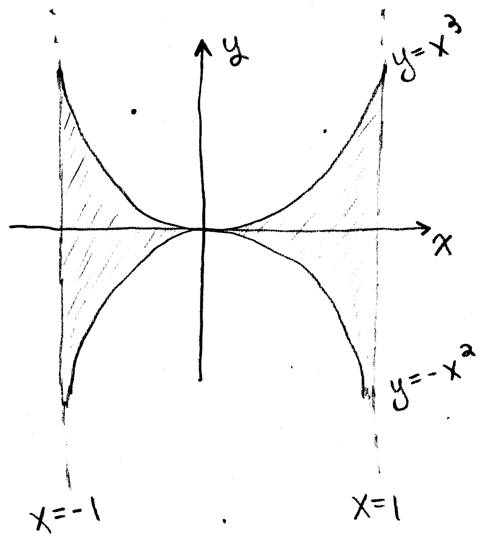
$\vec{\nabla} F(x_0,y_0,z_0)$ IS NORMAL TO GRAPH OF $z = f(x,y)$ AT (x_0,y_0,z_0).

3D VECTOR!

3. (6 points) Evaluate

$$\iint_P (x^2 - y)\, dA,$$

where P is the plane region between the parabolas $y = x^2$ and $y = -x^2$ on the interval $-1 \le x \le 1$. Evaluate the integral by hand, showing all work.



$$\int_{-1}^{1} \int_{-x^2}^{x^2} (x^2 - y)\, dy\, dx$$

$$= \int_{-1}^{1} x^2 y - \frac{1}{2} y^2 \Big|_{-x^2}^{x^2} dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{1}{2}x^4 + x^4 + \frac{1}{2}x^4\right) dx = \int_{-1}^{1} 2x^4\, dx$$

$$= \frac{2}{5} x^5 \Big|_{-1}^{1} = \boxed{\frac{4}{5}}$$

4. (4 points) The region P in the problem above is symmetric about both the x- and y-axes. Can you take advantage of these symmetries to simplify your integral? Explain.

THE INTEGRAND AND THE REGION ARE SYMMETRIC ABOUT $X=0$.

YES, $\iint_P (x^2 - y)\, dA = 2\int_0^1 \int_{-x^2}^{x^2} (x^2 - y)\, dy\, dx.$

THE INTEGRAND IS NOT SYMMETRIC ABOUT $y = 0$.

NO, $\iint_P (x^2 - y)\, dA \ne 2\int_{-1}^{1} \int_{0}^{x^2} (x^2 - y)\, dy\, dx.$

$$\sin\left(\frac{\pi}{2}\cos\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

5. (6 points) Find an equation of the plane tangent to the graph of $z = \sin(x\cos y)$ at the point where $(x, y) = (\pi/2, \pi/3)$.

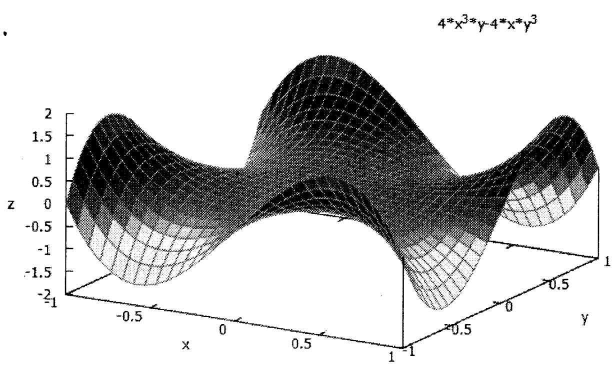
$$F(x,y,z) = \sin(x\cos y) - z$$

$$\vec{\nabla} F(x,y,z) = \cos(x\cos y)\left(\cos y\,\hat{\imath} - x\sin y\,\hat{\jmath}\right) - \hat{k}$$

$$n = \vec{\nabla} F\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{2}\hat{\imath} - \frac{\pi}{2}\cdot\frac{\sqrt{3}}{2}\hat{\jmath}\right) - \hat{k} = \frac{\sqrt{2}}{4}\hat{\imath} - \frac{\pi\sqrt{6}}{8}\hat{\jmath} - \hat{k}$$

$$\frac{\sqrt{2}}{4}\left(x - \frac{\pi}{2}\right) - \frac{\pi\sqrt{6}}{8}\left(y - \frac{\pi}{3}\right) - 1\left(z - \frac{\sqrt{2}}{2}\right) = 0$$

6. (4 points) The graph of $f(x, y) = 4x^3y - 4xy^3$ is shown below. f has a single critical point. Find it. Then use the graph to classify the critical point.



$f_x = 12x^2y - 4y^3 = 0 \qquad 4y(3x^2 - y^2) = 0$

$f_y = 4x^3 - 12xy^2 = 0$

$y = 0$ or $y^2 = 3x^2$

$y = 0 \to 4x^3 = 0 \to x = 0$

$y^2 = 3x^2 \to 4x^3 - 36x^3 = 0 \to x = 0 \to y = 0$

ONLY CRIT PT IS $(0,0)$. IT IS NEITHER A MAX NOR A MIN.

$\Rightarrow$ $(0,0,0)$ IS A SADDLE PT.

7. (8 points) Find and classify the critical points of $h(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y$. Find all relative extreme values.

$h_x: \quad 5y - 14x + 3 = 0 \Rightarrow -14x + 5y = -3$

$h_y: \quad 5x - 2y - 6 = 0 \Rightarrow 5x - 2y = 6$

$$\begin{aligned} -28x + 10y &= -6 \\ 25x - 10y &= 30 \\ \hline -3x &= 24 \end{aligned}$$

$x = -8$

$-2y = 6 + 40$

$\Rightarrow y = -23$

$(-8, -23)$ IS THE ONLY CRIT PT.

2ND PARTIALS...

$$D = \begin{vmatrix} -14 & 5 \\ 5 & -2 \end{vmatrix} = 28 - 25 = 3$$

$D > 0$ AND $h_{xx}(-8, -23) < 0 \Rightarrow$

$h(-8, -23) = 57$ IS A REL. MAX.

8. (10 points) Use Lagrange multipliers to find the extreme values of $f(x, y, z) = 2x + 3y + 5z$ subject to $x^2 + y^2 + z^2 = 19$.

$$\left.\begin{aligned} 2 &= \lambda 2x \\ 3 &= \lambda 2y \\ 5 &= \lambda 2z \end{aligned}\right\} \quad \begin{aligned} x &= 1/\lambda \\ y &= 3/2\lambda \\ z &= 5/2\lambda \end{aligned}$$

$$x^2 + y^2 + z^2 = 19$$

$$\frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{25}{4\lambda^2} = 19$$

$$\frac{38}{4\lambda^2} = 19 \quad \Rightarrow \quad 4\lambda^2 = 2$$

$$\lambda = \pm \frac{1}{\sqrt{2}}$$

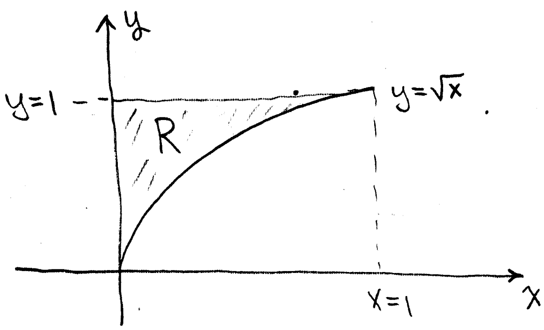
$$x = \pm \sqrt{2}, \quad y = \pm \frac{3\sqrt{2}}{2}, \quad z = \pm \frac{5\sqrt{2}}{2}$$

$$f\left(\sqrt{2}, \frac{3\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right) = 2\sqrt{2} + \frac{9\sqrt{2}}{2} + \frac{25\sqrt{2}}{2} = 19\sqrt{2} \text{ IS THE MAX.}$$

$$f\left(-\sqrt{2}, -\frac{3\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right) = -19\sqrt{2} \text{ IS THE MIN.}$$

9. (12 points) Sketch the region of integration, reverse the order of integration, and evaluate your new iterated integral (by hand, showing all work).

$$\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy\, dx$$



$$\int_{y=0}^{y=1} \int_{x=0}^{x=y^2} \sin\left(\frac{y^3+1}{2}\right) dx\, dy$$

$$= \int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy$$

$$u = \frac{y^3+1}{2}$$

$$du = \frac{3y^2}{2} dy$$

$$\frac{2}{3} du = y^2 dy$$

$$\frac{2}{3}\int_{1/2}^{1} \sin u \, du =$$

$$\frac{2}{3}\left(-\cos u\right)\Big|_{1/2}^{1}$$

$$= \frac{2}{3}\left(\cos\frac{1}{2} - \cos 1\right)$$

$$\approx 0.225$$

Math 173 - Test 3b
April 24, 2014

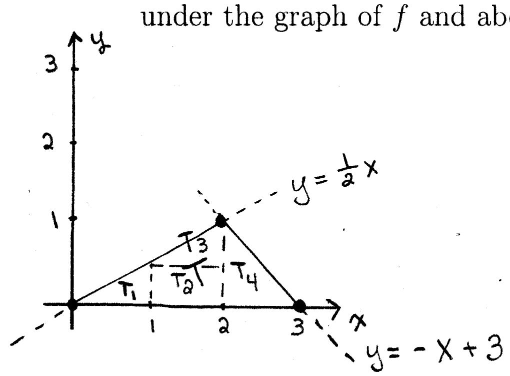
Name key

Score

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due Monday, April 30. You must work individually on this test.

1. (10 points) Let $f(x, y) = x + y + 1$ and let T be the triangle in the plane with vertices $(0, 0)$, $(2, 1)$, and $(3, 0)$.

   (a) Use a Riemann sum over 4 subregions to estimate the volume of the space region under the graph of f and above T.



Point in T_1 : $\left(\frac{1}{2}, \frac{1}{8}\right)$

Point in T_2 : $\left(\frac{3}{2}, \frac{1}{8}\right)$

Point in T_3 : $\left(\frac{3}{2}, \frac{5}{8}\right)$

Point in T_4 : $\left(\frac{5}{2}, \frac{1}{8}\right)$

Area of $T_1 = \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$

Area of $T_2 = 1\left(\frac{1}{2}\right) = \frac{1}{2}$

Area of $T_3 = \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{4}$

Area of $T_4 = \frac{1}{2}(1)(1) = \frac{1}{2}$

Riemann Sum $= f\left(\frac{1}{2}, \frac{1}{8}\right)\left(\frac{1}{4}\right) + f\left(\frac{3}{2}, \frac{1}{8}\right)\left(\frac{1}{2}\right) + f\left(\frac{3}{2}, \frac{5}{8}\right)\left(\frac{1}{4}\right) + f\left(\frac{5}{2}, \frac{1}{8}\right)\left(\frac{1}{2}\right)$

$= \frac{69}{16} = 4.3125$

   (b) Find the volume of the space region by setting up and evaluating (by hand) an iterated integral.

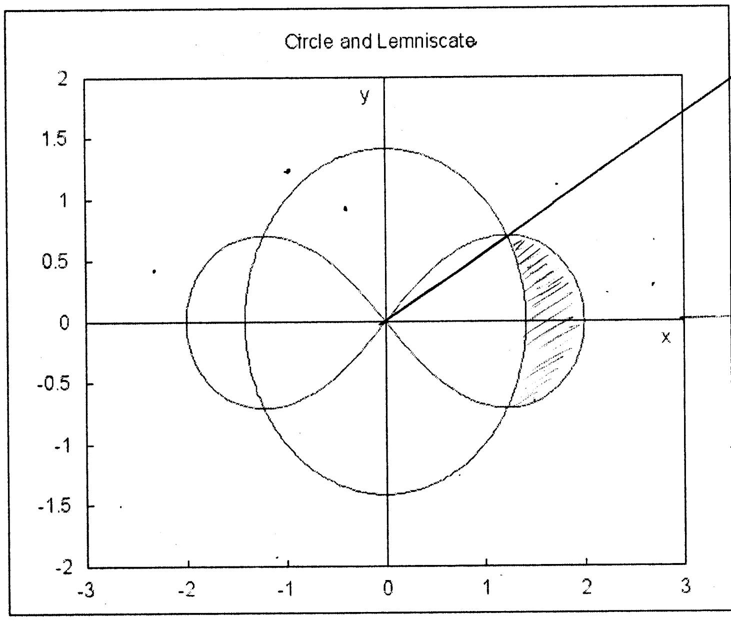
$$\int_{y=0}^{y=1} \int_{x=2y}^{x=3-y} (x + y + 1)\, dx\, dy = \int_0^1 \left(\frac{1}{2}x^2 + xy + x\right)\Big|_{x=2y}^{x=3-y} dy$$

$$= \int_0^1 \left(\frac{1}{2}(3-y)^2 + (3-y)y + 3 - y - \frac{1}{2}(2y)^2 - 2y^2 - 2y\right) dy$$

$$= \int_0^1 \left(\frac{9}{2} - 3y + \frac{1}{2}y^2 + 3y - y^2 + 3 - y - 2y^2 - 2y^2 - 2y\right) dy = \int_0^1 \left(\frac{15}{2} - 3y - \frac{9}{2}y^2\right)$$

$$= \frac{15}{2} - \frac{3}{2} - \frac{3}{2} = \frac{9}{2} = \boxed{4.5}$$

2. (8 points) Use a double integral to compute the area of the region in the right half-plane that lies outside the circle $r = \sqrt{2}$ and inside the lemniscate $r^2 = 4\cos 2\theta$.



$\theta = \frac{\pi}{6}$

$2 = 4\cos 2\theta$

$\Rightarrow \frac{1}{2} = \cos 2\theta$

$2\theta = \frac{\pi}{3} \quad \theta = \frac{\pi}{6}$

$\theta = 0.$

$$\text{Area} = 2\int_{\theta=0}^{\theta=\frac{\pi}{6}} \int_{r=\sqrt{2}}^{r=\sqrt{4\cos 2\theta}} r \, dr \, d\theta$$

$$= \int_0^{\pi/6} r^2 \Big|_{r=\sqrt{2}}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= \int_0^{\pi/6} (4\cos 2\theta - 2) \, d\theta$$

$$= 2\sin 2\theta - 2\theta \Big|_0^{\pi/6} = \sqrt{3} - \frac{\pi}{3}$$

$$\approx 0.68485$$

3. (12 points)

(a) Use Lagrange multipliers to find the point on the plane $x - 2y + 3z = 22$ that is closest to $(1, 2, 1)$.

Distance squared $= (x-1)^2 + (y-2)^2 + (z-1)^2$

Minimize $(x-1)^2 + (y-2)^2 + (z-1)^2$
s.t. $x - 2y + 3z = 22$

$2(x-1) = \lambda \Rightarrow x = 1 + \lambda/2$
$2(y-2) = -2\lambda \Rightarrow y = 2 - \lambda$
$2(z-1) = 3\lambda \Rightarrow z = 1 + 3\lambda/2$

$$1 + \frac{\lambda}{2} - 4 + 2\lambda + 3 + \frac{9\lambda}{2} = 22$$

$$7\lambda = 22 \Rightarrow \lambda = \frac{22}{7}$$

$x = \frac{18}{7}$
$y = -\frac{8}{7}$
$z = \frac{40}{7}$

This point must be closest. There is obviously not a farthest point.

(b) Find the distance from your solution in part (a) to the point $(1, 1, 1)$.

← Typo. Should be $(1,2,1)$.

$$\sqrt{\left(\frac{11}{7}\right)^2 + \left(-\frac{22}{7}\right)^2 + \left(\frac{33}{7}\right)^2} = \sqrt{\frac{242}{7}}$$

$$= \frac{11\sqrt{2}}{\sqrt{7}} \approx 5.8797$$

(c) Use the techniques of section 11.5 to find the distance from the plane to the point $(1, 2, 1)$.

$$\frac{|1 - 2(2) + 3(1) - 22|}{\sqrt{1+4+9}} = \frac{22}{\sqrt{14}} = \frac{11\sqrt{2}}{\sqrt{7}} \approx 5.8797$$

$$f(32,16) = 16(4) + 32(4) = 4 \times 48 = 192$$

4. (5 points) Find the linearization of $f(x, y) = yx^{2/5} + xy^{1/2}$ at the point $(32, 16)$. Then use your linearization to approximate $f(33, 15)$.

$$f_x(x,y) = \tfrac{2}{5} y x^{-3/5} + y^{1/2} \qquad f_x(32,16) = \tfrac{2}{5}(16)\left(\tfrac{1}{8}\right) + 4 = \tfrac{24}{5}$$

$$f_y(x,y) = x^{2/5} + \tfrac{1}{2} x y^{-1/2} \qquad f_y(32,16) = 4 + \tfrac{1}{2}(32)\left(\tfrac{1}{4}\right) = 8$$

$$\boxed{L(x,y) = 192 + \tfrac{24}{5}(x-32) + 8(y-16)}$$

$$f(33,15) \approx L(33,15) = 192 + \tfrac{24}{5} - 8 = 188\tfrac{4}{5} = \boxed{188.8}$$

ACTUAL VALUE OF $f(33,15) \approx 188.55$

5. (5 points) Find the absolute extreme values of $g(x, y) = x^2 + xy$ over the rectangle $R = \{(x, y) : -2 \le x \le 2,\ -1 \le y \le 1\}$.

INSIDE R:

$$\left.\begin{aligned} g_x(x,y) &= 2x + y = 0 \\ g_y(x,y) &= x = 0 \end{aligned}\right\} \ (x,y) = (0,0)$$

$$\begin{vmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$\boxed{(0,0,0) \text{ IS A SADDLE PT}}$

ALONG $x = 2$, $-1 \le y \le 1$:

$g(y) = 4 + 2y$

MAX IS $g(1) = 6$ ★

MIN IS $g(-1) = 2$

ALONG $x = -2$, $-1 \le y \le 1$:

$g(y) = 4 - 2y$

MIN IS $g(1) = 2$

MAX IS $g(-1) = 6$ ★

ALONG $y = -1$, $-2 \le x \le 2$:

$g(x) = x^2 - x = x(x-1)$ PARABOLA, OPENS UP, VERTEX AT $x = \frac{1}{2}$

$g(\frac{1}{2}) = -\frac{1}{4}$ MIN ★★

$g(-2) = 6$ MAX ★

$g(2) = 2$

ALONG $y = 1$, $-2 \le x \le 2$:

$g(x) = x^2 + x = x(x+1)$ PARABOLA, OPENS UP, VERTEX AT $-\frac{1}{2}$

$g(-\frac{1}{2}) = -\frac{1}{4}$ MIN ★★

$g(-2) = 2$

$g(2) = 6$ MAX ★

MAX VALUE IS 6 AT $(2,1), (-2,-1)$

MIN VALUE IS $-1/4$ AT $(-\frac{1}{2},1), (\frac{1}{2},-1)$