Math 173 - Final Exam

May 16, 2018

Name: Key

Score

Show all work. Supply explanations when necessary. Each problem is worth 12 points (unless otherwise indicated).

1. (14 points) Evaluate the limit or show that it does not exist.

(a) $\lim_{(x,y)\to(2,2)} \frac{y^2-x^2}{7x-7y} = \lim_{(x,y)\to(2,2)} \frac{\cancel{(y-x)}(y+x)}{-7\cancel{(y-x)}} = \boxed{-\frac{4}{7}}$

(b) $\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+3y^2}$

Along $y=0$:

$\lim_{x\to 0} \frac{0}{3x^2} = 0$

Along $y=x$:

$\lim_{x\to 0} \frac{2x^2}{6x^2} = \frac{2}{6} = \frac{1}{3}$

Limit DNE.

2. Find an equation of the plane containing the points $(3,-2,-2)$, $(5,0,2)$, and $(0,-4,-3)$.

P Q R

$\vec{PQ} = 2\hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{PR} = -3\hat{i} - 2\hat{j} - \hat{k}$

$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 4 \\ -3 & -2 & -1 \end{vmatrix}$

$= \hat{i}(6) - \hat{j}(10) + \hat{k}(2)$

$\vec{n} = 6\hat{i} - 10\hat{j} + 2\hat{k}$

I'll use

$\vec{n} = 3\hat{i} - 5\hat{j} + \hat{k}$

and point $Q(5,0,2)$.

Plane is

$3(x-5) - 5(y-0) + 1(z-2) = 0$

or

$\boxed{3x - 5y + z = 17}$

3. Find all critical points. Then use the 2nd partials test to determine whether each critical point is associated with a relative minimum, relative maximum, or saddle point.

$$f(x, y) = x^3 + y^3 - 3xy + 9$$

$f_x(x,y) = 3x^2 - 3y = 0 \Rightarrow y = x^2$

$f_y(x,y) = 3y^2 - 3x = 0$

$3x^4 - 3x = 0$

$3x(x^3 - 1) = 0$

$x = 0,\ x = 1$

$y = 0,\ y = 1$

$$D(x,y) = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$$

$D(0,0) = -9$

$\Rightarrow$ $(0,0,9)$ IS A SADDLE PT

$D(1,1) = 27 > 0$

& $f_{xx}(1,1) = 6 > 0$

$\Rightarrow$ $f(1,1) = 8$ IS A REL MIN

4. Let $\vec{u} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{w} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$.

(a) Find the angle between $\vec{u}$ and $\vec{w}$.

$\vec{u} \cdot \vec{\omega} = 3 + 2 + 2 = 7$

$\|\vec{u}\| = \sqrt{9+1+4} = \sqrt{14}$

$\|\vec{\omega}\| = \sqrt{1+4+1} = \sqrt{6}$

$\cos\theta = \dfrac{7}{\sqrt{14}\sqrt{6}}$

$\theta = \cos^{-1}\left(\dfrac{7}{\sqrt{14}\sqrt{6}}\right) \approx 40.2^\circ$

(b) Find the projection of $\vec{u}$ onto $\vec{w}$.

$$\text{proj}_{\vec{\omega}} \vec{u} = \frac{\vec{u} \cdot \vec{\omega}}{\vec{\omega} \cdot \vec{\omega}} \vec{\omega}$$

$$= \frac{7}{6}(\hat{\imath} + 2\hat{\jmath} - \hat{k}) = \frac{7}{6}\hat{\imath} + \frac{7}{3}\hat{\jmath} - \frac{7}{6}\hat{k}$$

5. Let $G(x, y, z) = x^2yz^3$.

(a) At the point $(1, 2, 1)$, in what direction does G increase the fastest?

$\nearrow$ DIRECTION OF $\vec{\nabla}G(1,2,1)$

$$\vec{\nabla}G(x,y,z) = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

$$\vec{\nabla}G(1,2,1) = 4\hat{i} + \hat{j} + 6\hat{k}$$

(b) Find the rate of change of G at the point $(1, 2, 1)$ in the direction $\vec{v} = \hat{\imath} + \hat{\jmath} + \hat{k}$.

$$\frac{1}{\|\vec{v}\|}\,\vec{\nabla}G(1,2,1)\cdot\vec{v} = \frac{1}{\sqrt{3}}(4+1+6) = \frac{11}{\sqrt{3}}$$

6. (15 points) Let P and Q be the points $(2, 1, -1)$ and $(0, -3, 2)$, respectively.

(a) Find a vector of length 5 in the direction of $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = -2\hat{i} - 4\hat{j} + 3\hat{k}$

$$\|\overrightarrow{PQ}\| = \sqrt{4+16+9} = \sqrt{29}$$

$$\frac{5}{\sqrt{29}}(-2\hat{i} - 4\hat{j} + 3\hat{k})$$

(b) Find a unit vector in the xy-plane that is orthogonal to $\overrightarrow{PQ}$.

$\hookrightarrow$ z-comp $= 0$

$\vec{u} = a\hat{i} + b\hat{j}$

$\vec{u}\cdot\overrightarrow{PQ} = -2a - 4b = 0$ LET'S USE $a = -2$, $b = 1$ $\rightarrow$ $-2\hat{i} + \hat{j}$

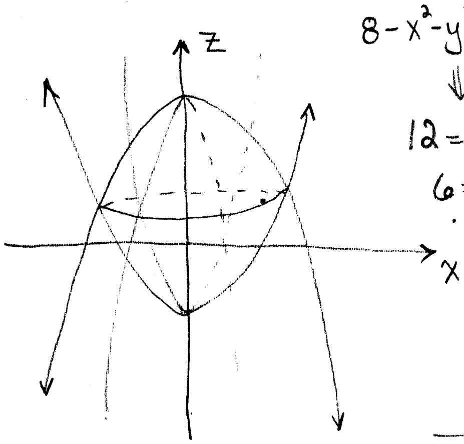
$$\frac{1}{\sqrt{5}}(-2\hat{i} + \hat{j})$$

(c) Find a set of symmetric equations for the line through P and Q.

Using $P(2,1,-1)$ ...

$$\frac{x-2}{-2} = \frac{y-1}{-4} = \frac{z+1}{3} = t$$

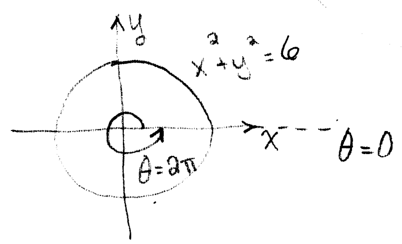
7. A solid lies between the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2 - 4$. The density of the solid at the point (x, y, z) is given by $\rho(x, y, z) = z + 5$. Set up the iterated integrals that give the mass of the solid and the moment about the xy-plane. Use your choice of coordinate systems. Do not evaluate your integrals.



$$8 - x^2 - y^2 = x^2 + y^2 - 4$$
$$\Downarrow$$
$$12 = 2x^2 + 2y^2$$
$$6 = x^2 + y^2$$

$$M = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{6}} \int_{z=r^2-4}^{z=8-r^2} (z+5)\, r \, dz\, dr\, d\theta$$

$$M_{xy} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{6}} \int_{z=r^2-4}^{z=8-r^2} z(z+5)\, r \, dz\, dr\, d\theta$$

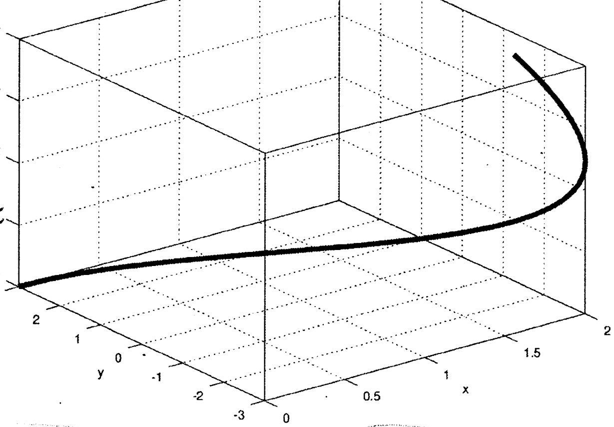


8. A particle is moving along the space curve described by

$$\vec{r}(t) = 2\sin t\,\hat{\imath} + 3\cos 2t\,\hat{\jmath} + t^2\,\hat{k}.$$

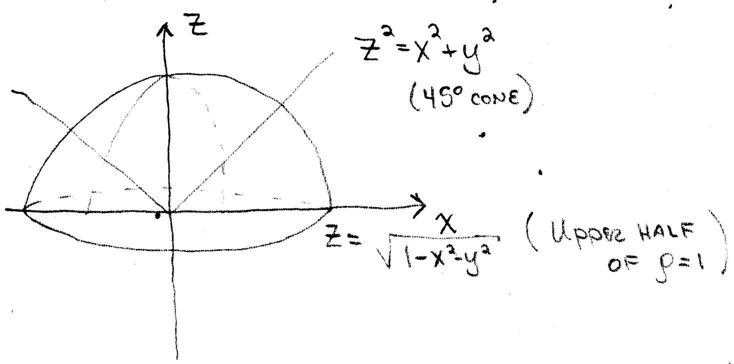
Set up the definite integral that gives the length of the curve from the point where $t = 0$ to the point where $t = 2$. Use your calculator to approximate the value of the integral. (Make sure your calculator is in radian mode.)

$$\vec{r}\,'(t) = 2\cos t\,\hat{\imath} - 6\sin 2t\,\hat{\jmath} + 2t\,\hat{k}$$
$$\Downarrow$$
$$\|\vec{r}\,'(t)\| = \sqrt{4\cos^2 t + 36\sin^2 2t + 4t^2}$$



$$\int_0^2 \sqrt{4\cos^2 t + 36\sin^2 2t + 4t^2}\, dt \approx 9.1856$$

9. (13 points) Use spherical coordinates to find the volume of the space region inside the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$ and below the cone $z^2 = x^2 + y^2$.



$$\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=\pi/4}^{\varphi=\pi/2} \int_{\rho=0}^{\rho=1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{2\pi}{3} \int_{\varphi=\frac{\pi}{4}}^{\varphi=\frac{\pi}{2}} \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3} \cos\varphi \Big|_{\pi/2}^{\pi/4}$$

$$= \frac{2\pi}{3} \cos\frac{\pi}{4} = \frac{\sqrt{2}\,\pi}{3}$$

10. Evaluate $\int_C (xz + 2y)\, ds$, where C is the line segment from $\underbrace{\overset{P}{(0,1,0)} \text{ to } \overset{Q}{(1,0,2)}}_{\vec{PQ} = \hat{\imath} - \hat{\jmath} + 2\hat{k}}$.

$$\int_C (xz + 2y)\, ds$$

$$C: \begin{cases} x = t \\ y = -t + 1, \quad 0 \le t \le 1 \\ z = 2t \end{cases}$$

$$\vec{r}(t) = t\,\hat{\imath} + (1-t)\,\hat{\jmath} + 2t\,\hat{k}$$

$$\vec{r}\,'(t) = \hat{\imath} - \hat{\jmath} + 2\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{1+1+4} = \sqrt{6}$$

$$= \int_0^1 \left[2t^2 + 2(1-t)\right] \sqrt{6}\, dt$$

$$= 2\sqrt{6} \int_0^1 (t^2 - t + 1)\, dt$$

$$= 2\sqrt{6}\left(\frac{1}{3} - \frac{1}{2} + 1\right)$$

$$= \frac{5\sqrt{6}}{3} \approx 4.0825$$

11. The quarterback of a football team throws the ball with an initial speed of 54 feet per second, at an angle of 35°, and at a height of 7 feet above the playing field. How far downfield has the ball traveled at the moment when it reaches its maximum height? (Ignore air resistance and use $g = 32\,\text{ft/sec}^2$.)

$$\vec{r}(t) = 54\cos 35^\circ\, t\,\hat{\imath} + \left(-16t^2 + 54\sin 35^\circ\, t + 7\right)\hat{\jmath}$$

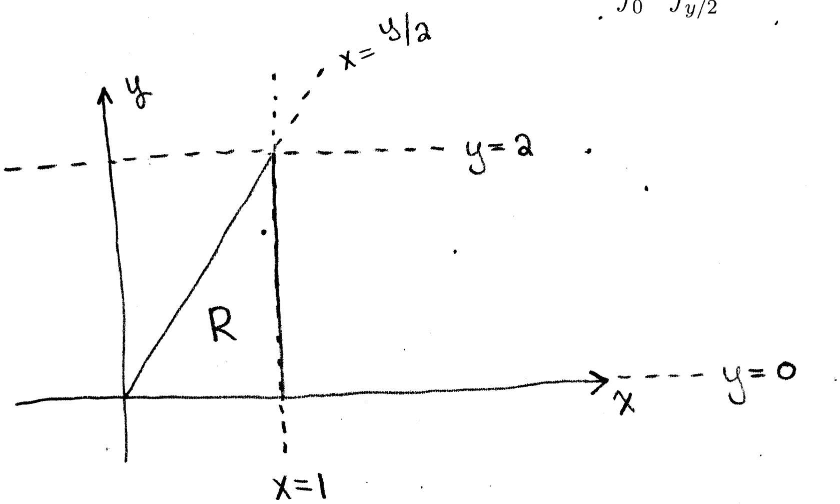
$$\vec{r}\,'(t) = 54\cos 35^\circ\,\hat{\imath} + \underbrace{\left(-32t + 54\sin 35^\circ\right)}_{\text{ZERO AT MAX...}}\hat{\jmath}$$

$$t = \frac{54\sin 35^\circ}{32} \approx 0.96791$$

$$54\cos 35^\circ\left(\frac{54\sin 35^\circ}{32}\right) \approx 42.815 \text{ FT}$$

12. Sketch the region of integration, reverse the order of integration, and evaluate the iterated integral by hand.

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$



$$\int_{x=0}^{x=1} \int_{y=0}^{y=2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 y e^{x^2} \Big|_{y=0}^{y=2x} dx$$

$$= \int_0^1 2x e^{x^2}\, dx = \int_{u=0}^{u=1} e^u\, du$$

$u = x^2$

$du = 2x\,dx$

$$= \boxed{e - 1 \approx 1.71828}$$