Math 173 - Extra Credit

April 1, 2019

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. This problem is worth 2 extra credit points. It is due Monday, April 8, 2019.

The gradient of $z = f(x, y)$ is defined in rectangular coordinates by

$$\vec{\nabla} f(x, y) = \frac{\partial z}{\partial x}\hat{\imath} + \frac{\partial z}{\partial y}\hat{\jmath}.$$

How does this look for a function in polar coordinates? To find out, we must write $\partial z/\partial x$ and $\partial z/\partial y$ in polar coordinates.

Suppose $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$.

1. Use the chain rule to write formulas for $\partial z/\partial r$ and $\partial z/\partial\theta$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

2. Solve your equations above for $\partial z/\partial x$ and $\partial z/\partial y$ in terms of r and θ.

Cramer's Rule...

$$\frac{\partial z}{\partial x} = \frac{\begin{vmatrix} \partial z/\partial r & \sin\theta \\ \partial z/\partial\theta & r\cos\theta \end{vmatrix}}{\underbrace{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}}_{r}} = \cos\theta\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial\theta}$$

$$\frac{\partial z}{\partial y} = \frac{\begin{vmatrix} \cos\theta & \partial z/\partial r \\ -r\sin\theta & \partial z/\partial\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}} = \frac{1}{r}\cos\theta\frac{\partial z}{\partial\theta} + \sin\theta\frac{\partial z}{\partial r}$$

3. Now write the formula for the gradient of a polar function.

$$f(x,y) = f(r\cos\theta, r\sin\theta) = F(r,\theta)$$

$$\vec{\nabla} F(r,\theta) = \left(\cos\theta\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial\theta}\right)\hat{\imath} + \left(\frac{1}{r}\cos\theta\frac{\partial z}{\partial\theta} + \sin\theta\frac{\partial z}{\partial r}\right)\hat{\jmath}$$