Math 173 - Test 1

February 14, 2019

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (4 points) Find a vector of magnitude 7 that is parallel to $\underbrace{-2\hat{\imath} - 6\hat{\jmath} + 12\hat{k}}_{\vec{u}}$.

$$\|\vec{u}\| = \sqrt{4+36+144} = \sqrt{184}$$

$$\pm \frac{7}{\|\vec{u}\|}\vec{u} = \boxed{\pm \frac{1}{\sqrt{184}}\left(14\hat{\imath} + 42\hat{\jmath} - 84\hat{k}\right)}$$

2. (5 points) Let β be the angle that $\vec{w} = -3\hat{\imath} + 5\hat{\jmath} - \hat{k}$ makes with the positive y-axis. Find the measure of β. Give your final answer in degrees, rounded to the nearest hundredth.

$$\|\vec{w}\| = \sqrt{9+25+1} = \sqrt{35}$$

$$\cos\beta = \frac{5}{\sqrt{35}}$$

$$\Rightarrow \boxed{\beta \approx 32.31^\circ}$$

3. (8 points) Forces with magnitudes 500 pounds and 200 pounds act on a hitch at angles of 30° and −45°, respectively, with the x-axis. Find the direction and magnitude of the resultant force.

$$\vec{F}_1 = 500\cos 30^\circ\,\hat{\imath} + 500\sin 30^\circ\,\hat{\jmath} = 250\sqrt{3}\,\hat{\imath} + 250\hat{\jmath}$$

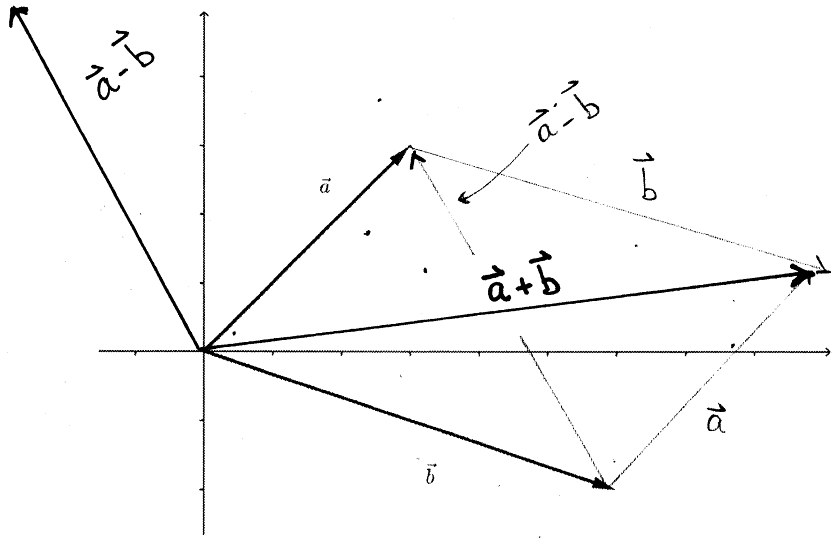
$$\vec{F}_2 = 200\cos(-45^\circ)\,\hat{\imath} + 200\sin(-45^\circ)\,\hat{\jmath} = 100\sqrt{2}\,\hat{\imath} - 100\sqrt{2}\,\hat{\jmath}$$

$$\vec{F}_1 + \vec{F}_2 = (250\sqrt{3} + 100\sqrt{2})\hat{\imath} + (250 - 100\sqrt{2})\hat{\jmath}$$

$$\approx 574.43\hat{\imath} + 108.58\hat{\jmath}$$

$$\|\vec{F}_1 + \vec{F}_2\| \approx 584.61, \quad \theta = \tan^{-1}\left(\frac{250-100\sqrt{2}}{250\sqrt{3}+100\sqrt{2}}\right) \approx 10.7^\circ$$

4. (7 points) The vectors $\vec{a}$ and $\vec{b}$ are shown below. Sketch the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.



5. (6 points) Consider the two points $P(2,2,3)$ and $Q(4,-5,9)$.

(a) Find the midpoint of the segment $\overline{PQ}$.

$$\left(\frac{2+4}{2}, \frac{2+(-5)}{2}, \frac{3+9}{2}\right)$$

$$= \boxed{\left(3, -\frac{3}{2}, 6\right)}$$

(b) Using the midpoint (above) as your initial point, find a set of parametric equations for the line segment $\overline{PQ}$.

$\overrightarrow{PQ} = 2\hat{i} - 7\hat{j} + 6\hat{k}$

POINT $\left(3, -\frac{3}{2}, 6\right)$

$$\boxed{\begin{aligned} x &= 3+2t \\ y &= -\frac{3}{2} - 7t \qquad -\frac{1}{2} \le t \le \frac{1}{2} \\ z &= 6+6t \end{aligned}}$$

6. (6 points) Use vectors to find the point that lies two-thirds of the way from P to Q.

$$P(4,3,0), \qquad Q(1,-3,3)$$

Another approach...

Let $O = (0,0,0)$ and compute

$$\vec{OP} + \frac{2}{3}\vec{PQ} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$$

$$\vec{PQ} = -3\hat{\imath} - 6\hat{\jmath} + 3\hat{k}$$

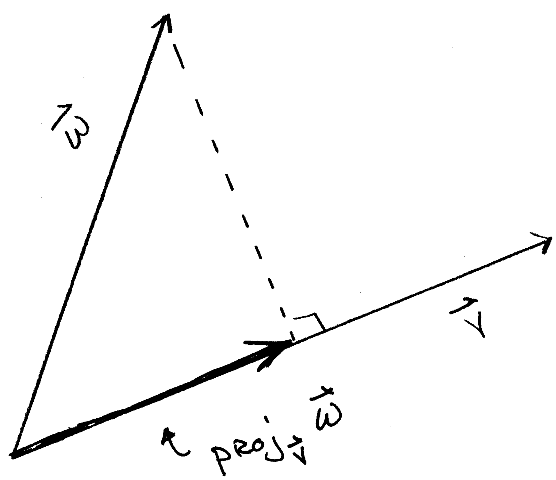
Segment $\overline{PQ}$:

$$x = 4 - 3t$$
$$y = 3 - 6t$$
$$z = 3t$$

$\frac{2}{3}$ of way from P to Q $\Rightarrow t = 2/3$

$$\boxed{x = 2, \; y = -1, \; z = 2}$$

7. (4 points) Sketch a diagram that shows two vectors, $\vec{v}$ and $\vec{w}$, and then show the vector $\text{proj}_{\vec{v}}\,\vec{w}$.



8. (5 points) Let $\vec{v} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$ and $\vec{w} = 7\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Compute $\text{proj}_{\vec{w}}\,\vec{v}$.

$$\text{proj}_{\vec{w}}\,\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\,\vec{w}$$

$$= \frac{21 - 5 - 4}{49 + 1 + 4}\,\vec{w} = \frac{12}{54}\,\vec{w} = \frac{2}{9}\,\vec{w}$$

$$= \frac{14}{9}\hat{\imath} + \frac{2}{9}\hat{\jmath} - \frac{4}{9}\hat{k}$$

9. (6 points) Find the area of the parallelogram determined by the vectors $\vec{x} = -3\hat{\imath}+4\hat{\jmath}+\hat{k}$ and $\vec{y} = -2\hat{\jmath} + 6\hat{k}$.

$$\vec{x} \times \vec{y} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix}$$

$$= \hat{\imath}(26) - \hat{\jmath}(-18) + \hat{k}(6)$$

$$= 26\hat{\imath} + 18\hat{\jmath} + 6\hat{k}$$

$$\text{AREA} = \|\vec{x} \times \vec{y}\|$$

$$= \sqrt{26^2 + 18^2 + 6^2}$$

$$= \sqrt{1036} \approx \boxed{32.187 \text{ Sq. UNITS}}$$

10. (4 points) Suppose $\vec{u} \times \vec{u} = \vec{0}$. Does this necessarily mean that $\vec{u} = \vec{0}$? Explain.

No, $\vec{u} \times \vec{u} = \vec{0}$ FOR ANY VECTOR $\vec{u}$.

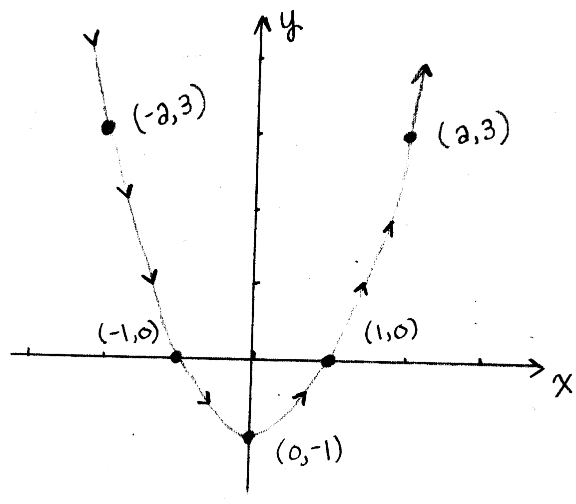
IN FACT, $\vec{a} \times \vec{b} = \vec{0}$ FOR ANY PARALLEL VECTORS $\vec{a}$ & $\vec{b}$.

11. (6 points) Determine a vector-valued function whose graph is the parabola described by $y = x^2 - 1$. Then sketch the graph and place arrows on the curve showing its orientation.

$x = t$

$y = t^2 - 1$

$$\boxed{\vec{r}(t) = t\hat{\imath} + (t^2-1)\hat{\jmath}}$$



SINCE $x = t$,

ORIENTATION IS

TOWARD POS X.

(COUNTER CLOCKWISE)

12. (10 points) Find an equation of the plane that contains the lines described by

$$(1,4,0) \quad \frac{x-1}{-2} = \frac{y-4}{1} = \frac{z}{1} \quad \text{and} \quad \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}. \quad (2,1,2)$$

(Hint: The plane's normal vector is perpendicular to both lines.)

$\vec{u} = -2\hat{i} + \hat{j} + \hat{k}$ $\qquad$ $\vec{v} = -3\hat{i} + 4\hat{j} - \hat{k}$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix}$$

$$= \hat{i}(-5) - \hat{j}(2+3) + \hat{k}(-8+3) = -5\hat{i} - 5\hat{j} - 5\hat{k}$$

I'LL USE $\hat{i} + \hat{j} + \hat{k}$ AND POINT $(1,4,0)$.

$$\boxed{x + y + z = 5}$$

13. (5 points) Consider the vector-valued functions

$$\vec{r_1}(t) = 2\cos 3t\,\hat{\imath} - 2\sin 3t\,\hat{\jmath} + 4\hat{k} \quad \text{and} \quad \vec{r_2}(t) = -6\sin 3t\,\hat{\imath} - 6\cos 3t\,\hat{\jmath}.$$

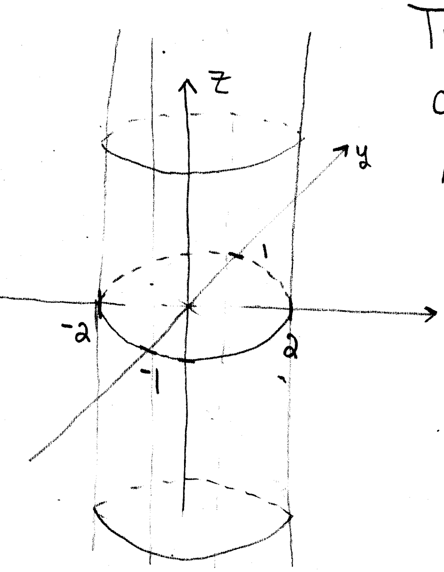
Show that $\vec{r_1}(t)$ and $\vec{r_2}(t)$ are orthogonal for any real number t.

$$\vec{r}_1(t) \cdot \vec{r}_2(t) = -12\cos 3t \sin 3t + 12 \sin 3t \cos 3t + 0$$
$$= 0$$

DOT PROD ZERO FOR ALL t

$\Rightarrow$ $\vec{r}_1(t)$ & $\vec{r}_2(t)$ ORTHOGONAL FOR ALL t.

14. (5 points) Describe (or sketch) the 3D surface defined by the equation $\dfrac{x^2}{4} + y^2 = 1$.



THE SURFACE IS AN ELLIPTICAL CYLINDER CENTERED ON AND PARALLEL TO THE Z-AXIS AND PASSING THROUGH THE ELLIPSE IN THE PLANE GIVEN BY $\frac{x^2}{4} + y^2 = 1$

15. (8 points) Find the measure of the acute angle between the planes given by

$$x - 2y + 2z = 2 \quad \text{and} \quad 5x + 3y - 2z = 0.$$

Give your final answer in degrees, rounded to the nearest hundredth.

$\vec{n}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{n}_2 = 5\hat{i} + 3\hat{j} - 2\hat{k}$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{5-6-4}{\sqrt{9}\sqrt{38}} = \frac{-5}{3\sqrt{38}}$$

$\theta \approx 105.69°$.

ACUTE ANGLE IS $180° - \theta \approx 74.31°$

16. (5 points) Find a 2D unit vector that is normal to the graph of $y = x^3$ at the point where $x = -2$. (Normal to the graph means perpendicular to the tangent line.)

$$\frac{dy}{dx} = 3x^2$$

$$m = \left.\frac{dy}{dx}\right|_{x=-2} = 12$$

$$m_\perp = -\frac{1}{12}$$

NORMAL VECTOR: $12\hat{i} - \hat{j}$.

MAG $= \sqrt{144+1} = \sqrt{145}$

$$\boxed{\frac{1}{\sqrt{145}}(12\hat{i} - \hat{j})}$$

17. (6 points) Determine the distance from the point $(3, 2, 1)$ to the plane described by $4x + 3y + 2z = 1$.

$$\frac{|4(3) + 3(2) + 2(1) - 1|}{\sqrt{4^2 + 3^2 + 2^2}} = \boxed{\frac{19}{\sqrt{29}} \approx 3.5282}$$