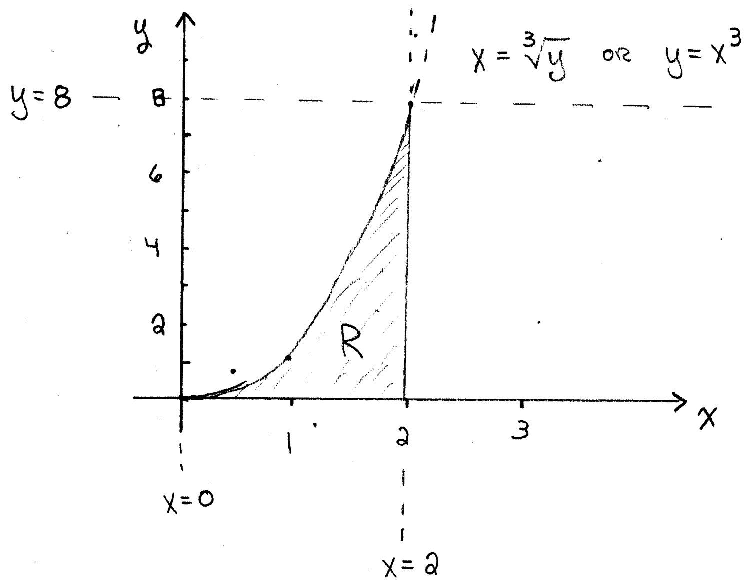
**Math 173 - Test 3a** Name key

May 2, 2019 Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Consider the iterated integral $\int_0^8 \int_{\sqrt[3]{y}}^{2} e^{x^4}\, dx\, dy$.

   (a) Sketch the region of integration.




   (b) Reverse the order of integration and evaluate by hand.

$$\int_{x=0}^{x=2} \int_{y=0}^{y=x^3} e^{x^4}\, dy\, dx = \int_0^2 y e^{x^4} \Big|_{y=0}^{y=x^3} dx$$

$$= \int_0^2 x^3 e^{x^4}\, dx = \frac{1}{4} \int_0^{16} e^u\, du = \frac{1}{4} e^u \Big|_0^{16}$$

$u = x^4$

$du = 4x^3\, dx$

$$= \frac{1}{4}\left(e^{16} - 1\right)$$

$\approx 2.2215 \times 10^6$

2. (6 points) Find an equation of the plane tangent to the graph of $z = \tan^{-1}(xy^2)$ at the point $(1, 1, \pi/4)$.

$$F(x,y,z) = \tan^{-1}(xy^2) - z$$

$$\vec{\nabla} F(x,y,z) = \frac{y^2}{1+(xy^2)^2}\,\hat{\imath} + \frac{2xy}{1+(xy^2)^2}\,\hat{\jmath} - \hat{k}$$

$$\vec{\nabla} F\left(1,1,\frac{\pi}{4}\right) = \frac{1}{2}\hat{\imath} + \hat{\jmath} - \hat{k}$$

I'LL USE

$$\vec{n} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$$

TAN. PLANE...

$$(x-1) + 2(y-1) - 2\left(z - \frac{\pi}{4}\right) = 0$$

OR

$$\boxed{x + 2y - 2z = 3 - \frac{\pi}{2}}$$

3. (4 points) Let $f(x,y) = x^2 + y^2 - 2x - 4y$. Find all points at which the greatest increase in the values of $f$ is in the direction of $\hat{\imath} + \hat{\jmath}$. (Your answer will be an entire collection of points that lie along a line.)

$$\vec{\nabla} f(x,y) = (2x-2)\hat{\imath} + (2y-4)\hat{\jmath}$$

$$= k(\hat{\imath} + \hat{\jmath}) \Rightarrow 2x - 2 = 2y - 4$$

$$\Rightarrow 2x - 2y = -2$$

$$\Rightarrow x - y = -1$$

$$\Rightarrow y = x + 1$$

$\boxed{\text{ALL POINTS OF THE FORM } (x, x+1).}$

4. (4 points) Suppose $w$ is a function of $x, y, z$ and $x, y, z$ are functions of $u, v$. Write the chain rule formulas for $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

5. (5 points) Near a buoy, the depth of a lake at the point $(x, y)$ is given by

$$d = 200 + 0.02x^2 - 0.001y^3,$$

where $x$, $y$, and $d$ are measured in meters. A fisherman is in a small boat starts at the point $(80, 60)$ and begins to move toward the buoy, which is located at $(0, 0)$. At what rate is the depth changing?

$\vec{v} = (0-80)\hat{i} + (0-60)\hat{j} = -80\hat{i} - 60\hat{j}$

I'll use $\vec{v} = -4\hat{i} - 3\hat{j}$, $\|\vec{v}\| = 5$

$\vec{\nabla} d = 0.04x\,\hat{i} - 0.003y^2\,\hat{j}$

$\vec{\nabla} d(80,60) = 3.2\hat{i} - 10.8\hat{j}$

$$D_{\vec{v}}\, d(80,60) = \frac{\vec{\nabla} d \cdot v}{\|\vec{v}\|} = \frac{1}{5}\left[(3.2)(-4) + (-10.8)(-3)\right] = 3.92$$

6. (7 points) Suppose $z$ is implicitly defined as a function of $x$ and $y$ by the equation $x \ln y + y^2 z + z^2 = 8$. Find $\partial z/\partial x$ and $\partial z/\partial y$.

$F(x,y,z) = x \ln y + y^2 z + z^2$

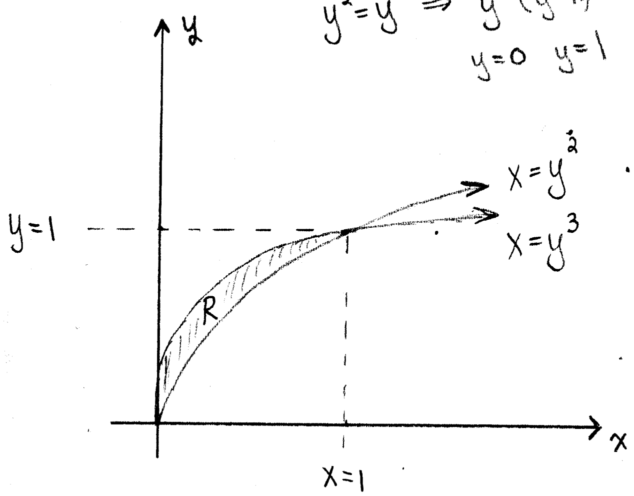
$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\ln y}{y^2 + 2z}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(x/y + 2yz)}{y^2 + 2z}$$

7. (8 points) Set up and evaluate the double integral that gives the volume of the space region under the surface described by $z = 2x + y^2$ and above the plane region in the 1st quadrant bounded by the graphs of $x = y^2$ and $x = y^3$.

$y^2 = y^3 \Rightarrow y^2(y-1) = 0$

$y=0 \quad y=1$




$$\iint_R (2x+y^2)\, dA = \int_{y=0}^{y=1} \int_{x=y^3}^{x=y^2} (2x+y^2)\, dx\, dy$$

$$= \int_0^1 x^2 + xy^2 \Big|_{x=y^3}^{x=y^2} dy$$

$$= \int_0^1 (y^4 + y^4 - y^6 - y^5)\, dy = \frac{1}{5} + \frac{1}{5} - \frac{1}{7} - \frac{1}{6}$$

$$= \frac{19}{210} \approx 0.090476$$

8. (6 points) Find and classify the critical points of $g(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$. Identify all relative extreme values and saddle points.

$g_x(x,y) = -10x + 4y + 16 = 0$

$g_y(x,y) = 4x - 2y = 0$

$-5x + 2y = -8$

$2x - y = 0$

$y = 2x$

$-5x + 4x = -8$

$-x = -8$

$x = 8, \; y = 16$

$(8,16)$ IS THE ONLY CRIT. PT.

$$D = \begin{vmatrix} -10 & 4 \\ 4 & -2 \end{vmatrix} = 20 - 16 = 4$$

$D > 0$ AND $g_{xx} < 0 \Rightarrow$ $g(8,16) = 74$ IS A RELATIVE MAX

**Math 173 - Test 3b**
May 2, 2019

Name key
Score

Show all work to receive full credit. Supply explanations where necessary. You must work individually on this test.

1. (10 points) Use Lagrange multipliers to find points on the sphere $x^2 + y^2 + z^2 = 9$ that are closest to and farthest from the point $(8, -8, 4)$.

MINIMIZE $d^2 = (x-8)^2 + (y+8)^2 + (z-4)^2$. (SQUARE OF DISTANCE)

s.t. $x^2+y^2+z^2 = 9$.

$$\begin{aligned} 2(x-8) &= \lambda 2x \\ 2(y+8) &= \lambda 2y \\ 2(z-4) &= \lambda 2z \end{aligned} \Rightarrow \begin{aligned} x-8 &= \lambda x \\ y+8 &= \lambda y \\ z-4 &= \lambda z \end{aligned} \Rightarrow \begin{aligned} x-\lambda x &= 8 \\ y-\lambda y &= -8 \\ z-\lambda z &= 4 \end{aligned}$$

$$x = \frac{8}{1-\lambda}, \quad y = \frac{-8}{1-\lambda}, \quad z = \frac{4}{1-\lambda}$$

$$x^2+y^2+z^2 = 9 \Rightarrow \frac{64+64+16}{(1-\lambda)^2} = 9 \qquad \frac{144}{9} = (1-\lambda)^2$$

$$16 = (1-\lambda)^2$$

$$\pm 4 = 1-\lambda \Rightarrow \lambda = -3, 5$$

$\lambda = -3$: $x = 2$, $y = -2$, $z = 1$

$\lambda = 5$: $x = -2$, $y = 2$, $z = -1$

$(2,-2,1) \Rightarrow d = \sqrt{36+36+9} = 9$ CLOSEST

$(-2,2,-1) \Rightarrow d = \sqrt{100+100+25} = 15$ FARTHEST

2. (10 points) Find and classify the critical points of $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$. Identify all relative extreme values and saddle points.

$f_x(x,y) = 6x^2 + y^2 + 10x = 0$

$f_y(x,y) = 2xy + 2y = 0 \Rightarrow 2y(x+1) = 0$

$y = 0$ OR $x = -1$

$6x^2 + 10x = 0$ $\qquad$ $6 + y^2 - 10 = 0$

$2x(3x+5) = 0$ $\qquad$ $y^2 = 4$

$x = 0$ OR $x = -\frac{5}{3}$ $\qquad$ $y = 2$ OR $y = -2$

$(0,0), \left(-\frac{5}{3}, 0\right)$ $\qquad$ $(-1, 2), (-1, -2)$

$$D(x,y) = \begin{vmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{vmatrix} = (12x+10)(2x+2) - 4y^2$$

$D(0,0) = 20$, $f_{xx}(0,0) = 10 \Rightarrow f(0,0) = 0$ IS A REL MIN.

$D\left(-\frac{5}{3}, 0\right) = 13\frac{1}{3}$, $f_{xx}\left(-\frac{5}{3}, 0\right) = -10 \Rightarrow f\left(-\frac{5}{3}, 0\right) = \frac{125}{27}$ IS A REL MAX.

$D(-1, 2) = -16 \Rightarrow (-1, 2, 3)$ IS A SADDLE POINT.

$D(-1, -2) = -16 \Rightarrow (-1, -2, 3)$ IS A SADDLE POINT.

3. (8 points) Use the definition of differentiability to show that $g(x,y) = 5y + 3xy - x^2$ is differentiable everywhere in $\mathbb{R}^2$.

$$\Delta z = g(x+\Delta x, y+\Delta y) - g(x,y) = \left[5(y+\Delta y) + 3(x+\Delta x)(y+\Delta y) - (x+\Delta x)^2\right] - \left[5y+3xy-x^2\right]$$

$$= 5y + 5\Delta y + 3xy + 3x\Delta y + 3y\Delta x + 3\Delta x\Delta y - x^2 - 2x\Delta x - \Delta x^2 - 5y - 3xy + x^2$$

$$= \underline{5\Delta y} + \underline{3x\Delta y} + \underline{3y\Delta x} + \underline{3\Delta x\Delta y} - \underline{2x\Delta x} - \underline{\Delta x^2}$$

$$= (3y-2x)\Delta x + (5+3x)\Delta y + (-\Delta x)\Delta x + (3\Delta x)\Delta y$$

$$= g_x(x,y)\,\Delta x + g_y(x,y)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

AND

$$\left.\begin{aligned}\epsilon_1 &= -\Delta x\\ \epsilon_2 &= 3\Delta x\end{aligned}\right\}\text{ GO TO ZERO AS } (\Delta x, \Delta y) \to (0,0).$$

TRUE FOR ALL $(x,y)$ IN $\mathbb{R}^2$. THEREFORE $g$ IS DIFFERENTIABLE EVERYWHERE.

4. (3 points) If without thinking carefully, you used a computer algebra system to evaluate each of these iterated integrals

$$\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx \qquad\qquad \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy,$$

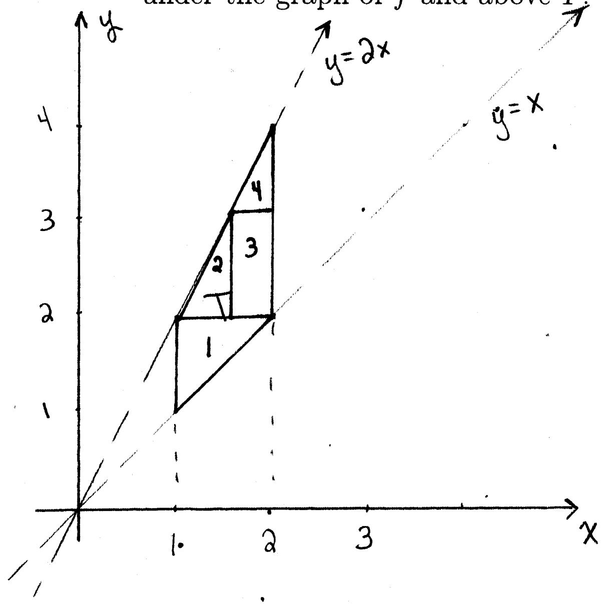
you would obtain different answers. How can this be so?

THE INTEGRAND $f(x,y) = \dfrac{x-y}{(x+y)^3}$ HAS AN INFINITE DISCONTINUITY AT $(0,0)$.

WE SHOULD NOT EXPECT THE INTEGRALS TO BE EQUAL OR TO NECESSARILY EXIST.

5. (12 points) Let $f(x, y) = \dfrac{y}{x^2 + y^2}$ and let $T$ be the trapezoid in the $xy$-plane bounded by the graphs of $y = x$, $y = 2x$, $x = 1$, and $x = 2$.

(a) Use a Riemann sum over 4 subregions to estimate the volume of the space region under the graph of $f$ and above $T$.




Region 1 : Area = $\frac{1}{2}(1)(1) = \frac{1}{2}$

Point = $(1.5, 1.75) = \left(\frac{3}{2}, \frac{7}{4}\right)$

Region 2 : Area = $\frac{1}{2}\left(\frac{1}{2}\right)(1) = \frac{1}{4}$

Point = $(1.25, 2.25) = \left(\frac{5}{4}, \frac{9}{4}\right)$

Region 3 : Area = $\left(\frac{1}{2}\right)(1) = \frac{1}{2}$

Point = $(1.75, 2.5) = \left(\frac{7}{4}, \frac{5}{2}\right)$

Region 4 : Area = $\frac{1}{2}\left(\frac{1}{2}\right)(1) = \frac{1}{4}$

Point = $(1.75, 3.25) = \left(\frac{7}{4}, \frac{13}{4}\right)$

$$\sum_{k=1}^{4} \frac{y_k}{x_k^2 + y_k^2} \Delta A_k = \left(\frac{28}{85}\right)\left(\frac{1}{2}\right) + \left(\frac{18}{53}\right)\left(\frac{1}{4}\right) + \left(\frac{40}{149}\right)\left(\frac{1}{2}\right) + \left(\frac{26}{109}\right)\left(\frac{1}{4}\right) \approx 0.44347$$

(b) Find the volume of the space region by setting up and evaluating (by hand) an iterated integral.

$$\int_{x=1}^{x=2} \int_{y=x}^{y=2x} \frac{y}{x^2+y^2}\, dy\, dx = \underbrace{\int_1^2 \frac{1}{2} \ln(x^2+y^2) \Big|_{y=x}^{y=2x} dx}_{\substack{u = x^2+y^2 \\ \frac{1}{2}du = y\,dy}}$$

$$= \int_1^2 \frac{1}{2}\Big[\underbrace{\ln(5x^2) - \ln(2x^2)}_{\ln\left(\frac{5x^2}{2x^2}\right)}\Big]\, dx$$

$$= \int_1^2 \frac{1}{2} \ln\frac{5}{2}\, dx = \frac{1}{2} \ln\frac{5}{2}\,(2-1)$$

$$= \boxed{\frac{1}{2} \ln\frac{5}{2}}$$

$\approx 0.45815$

6. (6 points) A surface described by the equation $y \ln xz^2 = 2$. Find a set of parametric equations for the line normal to the graph of the surface at the point $(e, 2, 1)$.

$$F(x,y,z) = y \ln xz^2 = y \ln x + 2y \ln z$$

$$\vec{\nabla} F(x,y,z) = \frac{y}{x}\hat{i} + (\ln x + 2 \ln z)\hat{j} + \frac{2y}{z}\hat{k}$$

$$\vec{n} = \vec{\nabla} F(e,2,1) = \frac{2}{e}\hat{i} + \hat{j} + 4\hat{k}$$

$$\boxed{\begin{aligned} x &= \frac{2}{e}t + e \\ y &= t + 2 \\ z &= 4t + 1 \end{aligned}}$$

7. (3 points) Suppose $z = f(x, y)$ is a differentiable function and $(x_0, y_0, z_0)$ is a point on its graph. Is it true that $\vec{\nabla} f(x_0, y_0)$ is normal to the graph of $f$ at $(x_0, y_0, z_0)$? Explain your reasoning.

NO, $\vec{\nabla} f(x_0, y_0)$ IS NORMAL TO THE CURVE $f(x,y) = z_0$ AT THE POINT $(x_0, y_0)$,

BUT $\vec{\nabla} f(x_0, y_0)$ IS NOT NORMAL TO THE SURFACE THAT IS THE GRAPH OF $z = f(x,y)$.

THE GRADIENT OF $F(x,y,z) = f(x,y) - z$ IS NORMAL TO THE GRAPH OF $f$ AT $(x_0, y_0, z_0)$.