Math 173 - Final Exam

May 15, 2019

Name key

Score

Show all work. Supply explanations when necessary. Each problem is worth 12 points (unless otherwise indicated).

1. (14 points) Determine each limit or explain why the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,2)} \frac{x+2y-4}{\sqrt{x+2y}-2}$ $\frac{0}{0}$ Form

$$\lim_{(x,y)\to(0,2)} \frac{x+2y-4}{\sqrt{x+2y}-2} \cdot \frac{\sqrt{x+2y}+2}{\sqrt{x+2y}+2} = \lim_{(x,y)\to(0,2)} \frac{\cancel{(x+2y-4)}(\sqrt{x+2y}+2)}{\cancel{(x+2y-4)}}$$

$$= \sqrt{4}+2 = \boxed{4}$$

(b) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sin(x^2-x+8y)}{24y+x^2-x}$

Along $x=1$: $\displaystyle\lim_{y\to 0} \frac{\sin 8y}{24y} = \frac{1}{3}\lim_{y\to 0}\frac{\sin 8y}{8y} = \frac{1}{3}$

Along $y=0$: $\displaystyle\lim_{x\to 1} \frac{\sin(x^2-x)}{x^2-x} = 1$

Limit DNE

2. Consider the plane passing through the points $P\,(1,2,3)$, $Q\,(3,-2,1)$, and $R\,(5,-1,-1)$.

   (a) Find a set of parametric equations for the line passing through P and normal to the plane.

$\overrightarrow{PQ} = 2\hat{i} - 4\hat{j} - 2\hat{k}$

$\overrightarrow{PR} = 4\hat{i} - 3\hat{j} - 4\hat{k}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & -2 \\ 4 & -3 & -4 \end{vmatrix} = \hat{i}(10) - \hat{j}(0) + \hat{k}(10) = 10\hat{i} + 10\hat{k}$$

I'll use $\vec{n} = \hat{i} + \hat{k}$

$$\boxed{\begin{aligned} x &= t+1 \\ y &= 2 \\ z &= t+3 \end{aligned}}$$

   (b) Determine the angle that the plane makes with the xy-plane. (The xy-plane has normal vector $\hat{k}$.)

$\vec{n}_1 = \hat{i} + \hat{k}$

$\vec{n}_2 = \hat{k}$

$$\cos\theta = \frac{(1)(0) + (0)(0) + (1)(1)}{\sqrt{1+1}\,\sqrt{1}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

3. A ball is kicked from ground level with an initial speed of 60 m/s. The ball travels a total horizontal distance of 200 m. What was the ball's initial angle and how long was its total flight? Ignore air resistance and use $g = 9.8\,\mathrm{m/s^2}$. (You'll probably also need to use $2\cos\theta\sin\theta = \sin 2\theta$.)

$$\vec{r}(t) = 60\cos\theta\, t\,\hat{i} + (-4.9t^2 + 60\sin\theta\, t)\,\hat{j}$$

$60\cos\theta\, t = 200$

$$\underbrace{-4.9t^2 + 60\sin\theta\, t = 0}$$

$t(-4.9t + 60\sin\theta) = 0$

$$t = \frac{60\sin\theta}{4.9}$$

$$\frac{60\cos\theta \cdot 60\sin\theta}{4.9} = 200$$

$$\underbrace{2\cos\theta\sin\theta} = \frac{(4.9)(200)}{(60)(30)}$$

$\sin 2\theta = 0.5\overline{4}$

$$\Rightarrow \boxed{\theta \approx 16.49^\circ}$$

$$t = \frac{60\sin\theta}{4.9} \approx \boxed{3.48 \text{ seconds}}$$

4. The position of a particle at time t is given by

$$\vec{r}(t) = 5\ln(1+t^2)\hat{\imath} + te^t\hat{\jmath} + (t^2+t+1)\hat{k}.$$

(a) At what time is the particle at the point $(5\ln 5, 2e^2, 7)$?

$\hookrightarrow te^t = 2e^2 \Rightarrow t = 2$

(b) Compute the unit tangent vector at the point $(5\ln 5, 2e^2, 7)$. (Plug in, then normalize.)

$$\vec{r}\,'(t) = \frac{10t}{1+t^2}\hat{\imath} + (e^t + te^t)\hat{\jmath} + (2t+1)\hat{k}$$

$$\vec{r}\,'(2) = 4\hat{\imath} + 3e^2\hat{\jmath} + 5\hat{k}$$

$$\hat{T}(2) = \frac{1}{\sqrt{9e^4+41}}(4\hat{\imath} + 3e^2\hat{\jmath} + 5\hat{k})$$

$$\|\vec{r}\,'(2)\| = \sqrt{9e^4+41}$$

(c) Find the projection of your unit tangent vector onto $\hat{\imath} + \hat{k}$.

$$\frac{\hat{T}(2)\cdot(\hat{\imath}+\hat{k})}{\|\hat{\imath}+\hat{k}\|^2}(\hat{\imath}+\hat{k}) = \frac{\frac{1}{\sqrt{9e^4+41}}\left(4(1)+5(1)\right)}{2}(\hat{\imath}+\hat{k})$$

$$= \frac{(9/2)}{\sqrt{9e^4+41}}(\hat{\imath}+\hat{k}) \approx 0.28(\hat{\imath}+\hat{k})$$

5. Find an equation of the plane tangent to the graph of $z = x^2\cos(\pi y) - \dfrac{8y^2}{x}$ at the point where $(x, y) = (2, -1)$.

$x = 2,\ y = -1,\ z = -4-4 = -8$

$$\underbrace{F(x,y,z) = x^2\cos(\pi y) - \frac{8y^2}{x} - z}_{\text{LEVEL SURFACE}} = 0$$

$$\vec{\nabla}F(x,y,z) = \left(2x\cos(\pi y) + \frac{8y^2}{x^2}\right)\hat{\imath} - \left(\pi x^2\sin(\pi y) + \frac{16y}{x}\right)\hat{\jmath} - \hat{k}$$

$$\vec{n} = \vec{\nabla}F(2,-1,-8) = (-4+2)\hat{\imath} + 8\hat{\jmath} - \hat{k} = -2\hat{\imath} + 8\hat{\jmath} - \hat{k}$$

TANGENT PLANE:

$$-2(x-2) + 8(y+1) - (z+8) = 0$$

OR

$$-2x + 8y - z = -4$$

6. Find the critical points of $g(x,y) = 2x^2 - 4xy + y^4 + 2$. Use the 2nd partials test to classify them, and identify all relative extreme values and saddle points.

$g_x(x,y) = 4x - 4y = 0 \Rightarrow x = y$

$g_y(x,y) = -4x + 4y^3 = 0 \Rightarrow -4y + 4y^3 = 0$

$4y(y^2-1) = 0$

$y=0,\ y=1,\ y=-1$

$x=0,\ x=1,\ x=-1$

$D(x,y) = \begin{vmatrix} 4 & -4 \\ -4 & 12y^2 \end{vmatrix} = 48y^2 - 16$

$D(0,0) = -16$

$\Rightarrow$ (0,0,2) IS A SADDLE POINT

$D(1,1) = 32,\ g_{xx}(1,1) > 0$

$\Rightarrow$ $g(1,1) = 1$ IS A REL MIN

$D(-1,-1) = 32,\ g_{xx}(-1,-1) > 0$

$\Rightarrow$ $g(-1,-1) = 1$ IS A REL MIN

7. The temperature distribution inside a containment vessel is given by

$$T(x,y,z) = 750(x^2+y^2)\cos\left(\frac{\pi z}{100}\right),$$

where the x, y, z are measured in centimeters and T is measured in degrees Celsius. Find the rate of change of temperature (directional derivative) at (10, 5, 50) in the direction toward the origin.

$\vec{\nabla} T(x,y,z) = 1500x \cos\frac{\pi z}{100}\,\hat{i} + 1500y \cos\frac{\pi z}{100}\,\hat{j} - \frac{750\pi}{100}(x^2+y^2)\sin\frac{\pi z}{100}\,\hat{k}$

$\vec{\nabla} T(10,5,50) = 0\hat{i} + 0\hat{j} - \frac{750\pi}{100}(125)(1)\hat{k}$

$= -937.5\pi\,\hat{k}$

$\vec{v} = (0-10)\hat{i} + (0-5)\hat{j} + (0-50)\hat{k}$

$= -10\hat{i} - 5\hat{j} - 50\hat{k}$

$$\frac{\vec{\nabla} T(10,5,50)\cdot\vec{v}}{\|\vec{v}\|} = \frac{(937.5\pi)(50)}{\sqrt{100+25+2500}} = \frac{46875\pi}{\sqrt{2625}} \approx \boxed{2874.26}$$

8. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$ subject to $x^2 + y^2 + z^2 = 36$.

① $0 = 2\lambda x \Rightarrow x = 0$ or $\lambda = 0$. ← CAN'T HAPPEN ③

② $2y = 2\lambda y \Rightarrow y = 0$ or $\lambda = 1$

③ $-10 = 2\lambda z$

④ $x^2 + y^2 + z^2 = 36$

$x=0,\ y=0,\ z = \pm 6$ | $x = 0,\ z = -5,\ y = \pm\sqrt{11}$

$(0,0,6)$ | $(0,\sqrt{11},-5)$

$(0,0,-6)$ | $(0,\sqrt{11},5)$

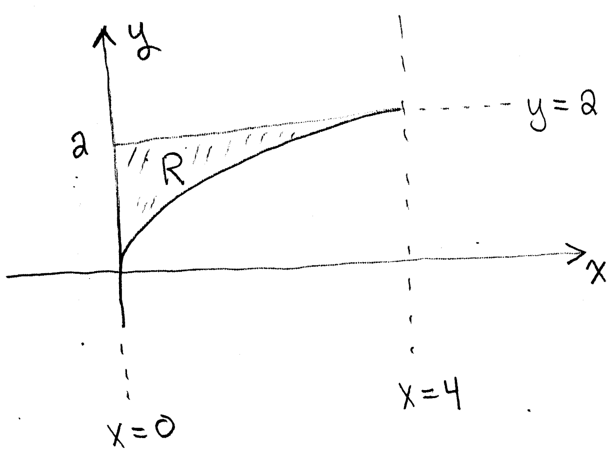
$f(0,0,6) = -60$ ← MIN

$f(0,0,-6) = 60$

$f(0,\sqrt{11},-5) = 61$
$f(0,-\sqrt{11},-5) = 61$ } MAX

9. Sketch the region of integration, reverse the order of integration, and evaluate the iterated integral by hand.

$$\int_0^4 \int_{\sqrt{x}}^2 \cos(y^3)\, dy\, dx$$



$$\int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \cos y^3 \, dx\, dy$$

$$= \int_0^2 y^2 \cos y^3 \, dy$$

$u = y^3$

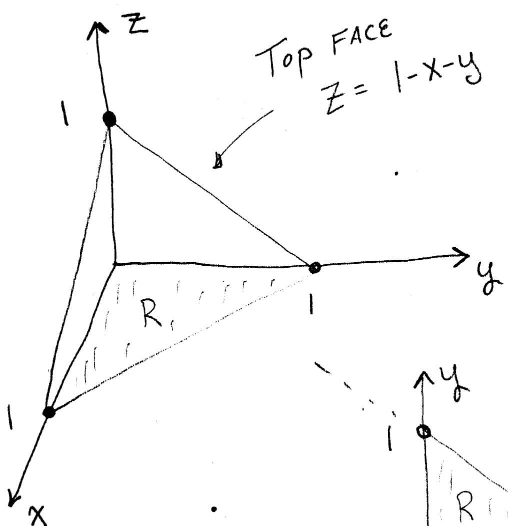
$\frac{1}{3} du = y^2 dy$

$$\frac{1}{3}\int_0^8 \cos u \, du = \boxed{\frac{\sin 8}{3}}$$

≈ 0.32979

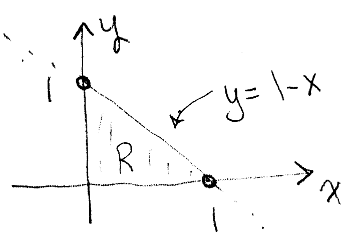
10. (16 points) A solid tetrahedron in the 1st octant is bounded by the coordinate planes and the plane $x + y + z = 1$. The density of the solid at (x, y, z) is given by $\rho(x, y, z) = 1 + z^2$.

(a) Set up the iterated integral that gives the total mass (M) of the solid. **Do not evaluate.**



$$M = \iiint_T (1+z^2)\, dV$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (1+z^2)\, dz\, dy\, dx$$



(b) Set up the iterated integral that gives the moment about the xy-plane (M_{xy}). **Do not evaluate.**

$$M_{xy} = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} z(1+z^2)\, dz\, dy\, dx$$

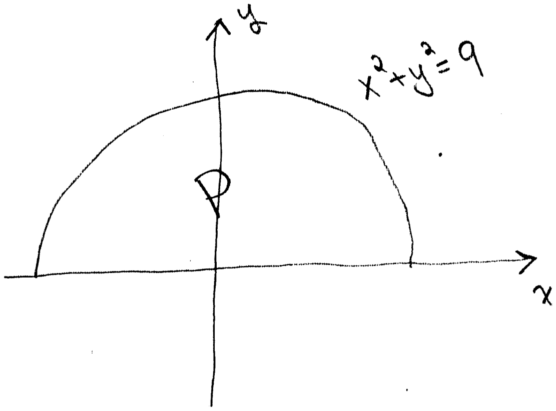
(c) Which coordinate of the center of mass would you have if you computed M_{xy}/M?

$\boxed{z\text{-COORD}}$ $\qquad \bar{z} = \dfrac{M_{xy}}{M}$

11. Evaluate

$$\iint_P e^{x^2+y^2}\, dA,$$

where P is the plane region above the x-axis and inside the circle $x^2+y^2=9$. Evaluate by hand, showing all work.



$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3} e^{r^2}\, r\, dr\, d\theta$$

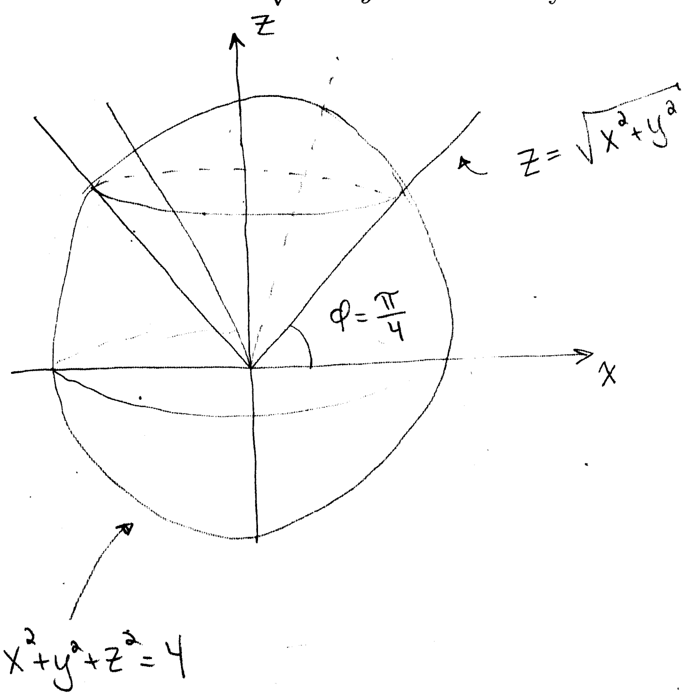
$$u=r^2$$

$$\tfrac{1}{2}\,du = r\,dr$$

$$\frac{1}{2}\int_0^{\pi}\int_0^{9} e^{u}\, du\, d\theta = \frac{\pi}{2}\, e^{u}\Big|_0^{9}$$

$$= \boxed{\frac{\pi}{2}\left(e^{9}-1\right)}$$

12. Find the volume of the solid inside the sphere $x^2+y^2+z^2=4$ but outside the cone $z=\sqrt{x^2+y^2}$. Evaluate by hand.



$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=\frac{\pi}{4}}^{\varphi=\pi}\int_{\rho=0}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 2\pi\left(\frac{8}{3}\right)\int_{\pi/4}^{\pi} \sin\varphi\, d\varphi$$

$$= \frac{16\pi}{3}\cos\varphi\,\Big|_{\pi}^{\pi/4}$$

$$= \boxed{\frac{16\pi}{3}\left(\frac{\sqrt{2}}{2}+1\right)}$$

$$\approx 4.9075$$