Solving First-Order, Linear ODEs

It is common in mathematics to multiply both sides of an equation by an expression that will in some way simplify the equation. In solving a first-order, linear ODE,

$$y'(x) + p(x)y(x) = q(x),$$

we will use the same idea. We seek an integrating factor, $\mu(x)$, such that

$$(\mu y)' = \mu q.$$

Let's see what $\mu(x)$ should look like...

We start with

$$y' + p y = q.$$

Multiply both sides by $\mu(x)$:

$$\mu y' + \mu p y = \mu q.$$

We want the left-hand side to be equal to $(\mu y)'$:

$$(\mu y)' = \mu y' + \mu' y = \mu y' + \mu p y.$$

It follows that

$$\mu' y = \mu p y$$
 or $\mu' = \mu p$.

And it is easy to see that the last equation is satisfied by

$$\mu(x) = e^{\int p(x) \, dx}.$$

Solving 1st order, linear ODEs

- Given y'(x) + p(x)y(x) = q(x)
- Find the integrating factor $\mu(x) = e^{\int p(x) dx}$
- The solution follows from

$$\mu(x) y(x) = \int \mu(x) q(x) dx$$

(Don't forget your constant of integration!)

• Or, if you are given an initial condition...

$$\mu(x) y(x) = \mu(x_0) y(x_0) + \int_{x_0}^x \mu(t) q(t) dt$$