

Math 216 - Quiz 1

September 1, 2010

Name key _____
Score _____

Show all work to receive full credit. Supply explanations when necessary.

1. (2 points) In radioactive decay, the number of nuclei, N , decays according to the differential equation

$$\frac{dN}{dt} = kN,$$

where k is the decay constant and t is time. We have already solved this DE to get the exponential growth/decay model $N(t) = N_0 e^{kt}$. Use this model to solve the following problem.

The half-life of carbon-14 is approximately 5700 years. One of the Dead Sea scrolls found in 1947 contained 76% of its initial amount of carbon-14. About how old was the scroll?

$$N(t) = C e^{kt}$$

SINCE $N(0) = C$,
C STANDS FOR THE
INITIAL AMOUNT.
I'LL CALL IN N_0 .

$$\Rightarrow N(t) = N_0 e^{kt}$$

HALF-LIFE = 5700

$$\Rightarrow \frac{1}{2} N_0 = N_0 e^{5700k}$$

$$k = \frac{\ln \frac{1}{2}}{5700}$$

$$\text{Solve } 0.76 N_0 = N_0 e^{kt}$$

$$0.76 = e^{kt}$$

$$\ln 0.76 = kt$$

$$\frac{\ln 0.76}{k} = t \Rightarrow t \approx 2256.8 \text{ YEARS}$$

2. (3 points) Consider the differential equation $y' = 1 - y^2$.

- (a) Use the attached Sage code or another online tool to plot the direction field for the DE. Without solving the DE, draw a rough sketch of the solution curve through $(0, 0)$. What are the limits of your solution as $x \rightarrow \pm\infty$?

SEE ATTACHED. $\lim_{x \rightarrow \infty} y(x) = 1$

$$\lim_{x \rightarrow -\infty} y(x) = -1$$

- (b) Solve the differential equation along with the initial condition $y(0) = 0$.

$$\frac{dy}{1-y^2} = dx$$

$$\frac{1}{2} \int \left(\frac{1}{1-y} + \frac{1}{1+y} \right) dy = \int dx$$

$$1+y = C_3 e^{2x} - y C_3 e^{2x}$$

$$\int \frac{1}{1-y^2} dy = \int dx$$

$$\frac{1}{2} \left[-\ln|1-y| + \ln|1+y| \right] = x + C_1$$

$$y = \frac{C_3 e^{2x} - 1}{1 + C_3 e^{2x}}$$

$$\frac{1}{1-y^2} = \frac{\frac{1}{2}}{1-y} + \frac{\frac{1}{2}}{1+y}$$

$$\ln \left| \frac{1+y}{1-y} \right| = 2x + C_2$$

$$y(0) = 0 = \frac{C_3 - 1}{1 + C_3}$$

$$\Rightarrow C_3 = 1$$

EASY PFD.

$$\frac{1+y}{1-y} = C_3 e^{2x}$$

$$y = \frac{e^{2x} - 1}{1 + e^{2x}}$$

$$m(0) = 0$$

3. (3 points) The slope m of a curve is 0 where the curve crosses the y -axis, and in general,

$$\frac{dm}{dx} = \sqrt{1+m^2}, \quad m(0) = 0.$$

Find m as a function of x . (In order to integrate, you'll need to use a trigonometric substitution. Show all work.)

$$\frac{dm}{\sqrt{1+m^2}} = dx$$

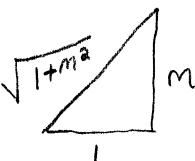
$$m = \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dm = \sec^2 \theta d\theta$$

$$\int \frac{\sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int dx$$

BUT ABS VALUE

NOT NECESSARY



$$\ln |\sqrt{1+m^2} + m| = x + C$$

THIS IS POSITIVE.

$$4. (2 \text{ points}) \text{ Solve: } \frac{dy}{dx} - \frac{y}{x} = x^2, \quad y(1) = 3$$

STANDARD FORM, LINEAR

$$\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{|x|} = \frac{1}{x} \quad \text{ASSUMING } x > 0$$

$$\frac{1}{x} y(x) = \int \frac{1}{x} x^2 dx$$

$$= \frac{x^2}{2} + C$$

$$y(x) = \frac{x^3}{2} + Cx$$

$$y(1) = 3 \Rightarrow \frac{1}{2} + C = 3 \Rightarrow C = \frac{5}{2}$$

$$y = \frac{x^3}{2} + \frac{5}{2}x$$

$$\sqrt{1+m^2} + m = C_2 e^x$$

$$m(0) = 0$$

$$\Rightarrow C_2 = 1$$

NOW SOLVE FOR m :

$$(\sqrt{1+m^2})^2 = (e^x - m)^2$$

$$1+m^2 = e^{2x} - 2me^x + m^2$$

$$m = \frac{e^{2x}-1}{2e^x}$$

Direction Fields

```
x,y = var('x,y')
f(x,y)=1-y^2
v = vector([1,f(x,y)])
u=v/v.norm()
p=plot_vector_field( u, (x,-10,10), (y,-5,5) )
p1=plot((exp(2*x)-1)/(1+exp(2*x)),(x,-10,10),rgbcolor=(1,0,0) )
show(p+p1)
```

