

**Math 216 - Test 1**  
September 22, 2010

Name key \_\_\_\_\_  
Score \_\_\_\_\_

Show all work (even on multiple choice problems). Supply explanations when necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

(a)  $(x^2 + y^2) dx + 2xy dy = 3 dx$

1<sup>ST</sup> ORDER, ORDINARY, NONLINEAR

(b)  $\frac{\partial^2 u}{\partial y^2} - xy^2 u = 6y^2 - \sin x$

2<sup>ND</sup> ORDER, PARTIAL, LINEAR

(c)  $z'' - 4z' + 3z = (t^2 + 5) \sin t$

2<sup>ND</sup> ORDER, ORDINARY, LINEAR

(d)  $yy''' + xy'' = xe^x$

3<sup>RD</sup> ORDER, ORDINARY, NONLINEAR

2. (2 points) A family of curves satisfies the differential equation

$$\frac{dy}{dx} = \frac{y}{x(1+x)}.$$

What is the differential equation for the family of orthogonal trajectories?

(a)  $\frac{dy}{dx} = \frac{x(1+x)}{y}$

(b)  $\frac{dy}{dx} = -\frac{x(1+x)}{y}$

(c)  $\frac{dy}{dx} = \frac{y(1-x)}{x}$

(d)  $\frac{dy}{dx} = -\frac{y}{x(1+x)}$

3. (4 points) Without solving the following differential equation, show that it is NOT exact.

$$\underbrace{(\cos x \sin x - xy^2)}_{M(x,y)} dx + \underbrace{y(1+x^2)}_{N(x,y)} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (\cos x \sin x - xy^2) = -2xy \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Not equal}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (y + yx^2) = 2xy \quad \text{Not exact}$$

4. (12 points) Solve the following initial value problem:

$$x \sqrt{1-y^2} dx = dy, \quad y(0) = 1/2$$

$$x dx = \frac{1}{\sqrt{1-y^2}} dy$$

$$y(0) = \frac{1}{2}$$

$$\Rightarrow \sin c = \frac{1}{2} \Rightarrow c = \frac{\pi}{6}$$

$$\frac{x^2}{2} + C = \sin^{-1} y$$

$$\text{Assuming } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$y = \sin \left( \frac{x^2}{2} + c \right)$$

$$y(x) = \sin \left( \frac{x^2}{2} + \frac{\pi}{6} \right)$$

5. (2 points) Suppose you are sketching the direction field for the differential equation

$$\frac{dy}{dx} = \frac{5x+10y}{-4x+3y}$$

What is the slope of the solution curve passing through  $(1, 2)$ ?

- (a) 12.5
- (b) 25
- (c) 0.25
- (d) 10

$$\left. \frac{dy}{dx} \right|_{(1,2)} = \frac{5(1)+10(2)}{-4(1)+3(2)} = \frac{25}{2}$$

6. (15 points) Solve the following initial value problem:

$$x^2y' + 4xy = 3x^2 + 6, \quad y(1) = 2$$

$$y' + \frac{4}{x}y = 3 + \frac{6}{x^2}, \quad \text{LET'S ASSUME } x > 0.$$

$$\mu(x) = e^{\int \frac{4}{x} dx} = e^{4 \ln x} = x^4$$

$$x^4 y = \int x^4 \left( 3 + \frac{6}{x^2} \right) dx$$

$$x^4 y = \int (3x^4 + 6x^2) dx$$

$$= \frac{3}{5}x^5 + 2x^3 + C$$

$$y(x) = \frac{3}{5}x + \frac{2}{x} + \frac{C}{x^4}$$

$$y(1) = 2 \Rightarrow \frac{3}{5} + 2 + C = 2$$

$$\Rightarrow C = -\frac{3}{5}$$

$$y(x) = \frac{3x}{5} + \frac{2}{x} - \frac{3}{5x^4}$$

7. (15 points) Solve the following Bernoulli equation:

$$\frac{dy}{dx} - y = e^x y^2$$

$$y^{-2} \frac{dy}{dx} - y^{-1} = e^x$$

$u = y^{-1}$  Assuming  $y \neq 0$  (But  $y(x) = 0$  is a solution)

$$\frac{du}{dx} = -y^{-2} \frac{dy}{dx}$$

Substitute...

$$-\frac{du}{dx} - u = e^x \Rightarrow \frac{du}{dx} + u = -e^x$$

$$\mu(x) = e^{\int dx} = e^x$$

$$e^x u = \int e^x (-e^x) dx$$

$$= - \int e^{2x} dx = -\frac{1}{2} e^{2x} + C$$

$$u = -\frac{1}{2} e^x + C e^{-x}$$

$$\text{But } u = \frac{1}{y}$$

$$y(x) = \frac{1}{Ce^{-x} - \frac{1}{2} e^x}$$

MULT BY  $\frac{2e^x}{2e^x}$

or

$$y(x) = \frac{2e^x}{C - e^{2x}}, \quad y(x) = 0$$

IS ALSO  
A SOL'N

8. (15 points) Consider the following differential equation:

$$(y^3 - y^2 \sin x - x) dx + (3xy^2 + 2y \cos x) dy = 0$$

(a) Show that the equation is exact.

$$\frac{\partial}{\partial y} (y^3 - y^2 \sin x - x) = 3y^2 - 2y \sin x = \frac{\partial}{\partial x} (3xy^2 + 2y \cos x)$$

(b) Solve the equation.

$$\frac{\partial F}{\partial x} = y^3 - y^2 \sin x - x \Rightarrow F(x,y) = xy^3 + y^2 \cos x - \frac{x^2}{2} + g(y)$$

$$\frac{\partial F}{\partial y} = 3xy^2 + 2y \cos x \Rightarrow F(x,y) = xy^3 + y^2 \cos x + h(y)$$

$$F(x,y) = xy^3 + y^2 \cos x - \frac{x^2}{2}$$

SOL'N IS

$$xy^3 + y^2 \cos x - \frac{x^2}{2} = C$$

(c) Is your solution implicit or explicit?

9. (10 points) Find the orthogonal trajectories for the family of curves described by the equation  $y = Ce^{-x}$ .

$\underbrace{\hspace{1cm}}$

$$e^x y = C$$

$$\frac{d}{dx}(e^x y) = 0$$

$$e^x y + e^x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -y$$

Ortho Traj's SATISFY

$$\frac{dy}{dx} = \frac{1}{y}$$

$$y dy = dx$$

$$\frac{y^2}{2} = x + C$$

$$y^2 = 2x + C$$

OR

$$y = \pm \sqrt{2x + C}$$

10. (5 points) The following equation is homogeneous. Make the appropriate substitution, separate the variables, and then STOP. Do not solve.

$$2x^3y dx + (x^4 + y^4) dy = 0$$

$$\frac{dy}{dx} = \frac{-2x^3y}{x^4 + y^4} \cdot \frac{1}{x^4}$$

$$u = \frac{y}{x}, \quad u + x \frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-2\frac{y}{x}}{1 + (\frac{y}{x})^4}$$

$$u + x \frac{du}{dx} = \frac{-2u}{1+u^4}$$

$$x \frac{du}{dx} = \frac{-2u}{1+u^4} - \frac{u(1+u^4)}{1+u^4}$$

$$\frac{1+u^4}{-3u-u^5} du = \frac{1}{x} dx$$

$$\frac{-3u-u^5}{1+u^4}$$

10. (10 points) The US population from 1790 to 1940 can be approximated by the solution of the initial value problem

$$\frac{dP}{dt} = 0.0318P - 0.000170P^2, \quad P(0) = 3.9,$$

BERNoulli  
OR SEPARABLE

where  $P$  is in millions and  $t$  is in years since 1790.

(a) Solve for  $P(t)$ .

$$\frac{dp}{dt} - 0.0318p = -0.00017p^2$$

$$\frac{1}{p^2} \frac{dp}{dt} - 0.0318p^{-1} = -0.00017$$

$$u = p^{-1}$$

$$\frac{du}{dt} = -1p^{-2} \frac{dp}{dt}$$

$$-\frac{du}{dt} - 0.0318u = -0.00017$$

$$\frac{du}{dt} + 0.0318u = 0.00017$$

$$\mu(t) = e^{\int 0.0318 dt} = e^{0.0318t}$$

$$e^{0.0318t} u = \int 0.00017 e^{0.0318t} dt$$

$$e^{0.0318t} u = \frac{0.00017}{0.0318} e^{0.0318t} + C$$

$$u = \frac{0.00017}{0.0318} + Ce^{-0.0318t}$$

$$P(t) = \frac{1}{0.005346 + Ce^{-0.0318t}}$$

$$P(0) = 3.9 \Rightarrow 3.9 = \frac{1}{0.005346 + C}$$

$$\Rightarrow C = 0.25106$$

$$P(t) = \frac{1}{0.005346 + 0.25106 e^{-0.0318t}}$$

- (b) If the actual population in 1900 was 76.0 million, find the percent error in the approximation given by  $P$ .

IN 1900,  $t = 110$ .

$$P(110) = 77.3 \text{ million}$$

$$\frac{77.3 - 76}{76}$$

$$\% \text{ error} = \frac{77.3 - 76}{76}$$

$$\approx 0.017 = 1.7\%$$