

Math 216 - Test 2a
October 27, 2010

Name key _____
Score _____

Show all work. Supply explanations when necessary.

1. (10 points) Solve: $y'' + 2y' - 8y = 0$; $y(0) = 2$, $y'(0) = 10$

Char eq is $r^2 + 2r - 8 = 0$

$$(r+4)(r-2) = 0$$

$$r = -4, r = 2$$

Fun sol'n set is $\{e^{-4x}, e^{2x}\}$

$$y(x) = c_1 e^{-4x} + c_2 e^{2x}$$

$$y(0) = 2 \Rightarrow 2 = c_1 e^0 + c_2 e^0$$

$$\Rightarrow 2 = c_1 + c_2$$

$$y'(x) = -4c_1 e^{-4x} + 2c_2 e^{2x}$$

$$y'(0) = 10 \Rightarrow 10 = -4c_1 e^0 + 2c_2 e^0$$

$$\Rightarrow 10 = -4c_1 + 2c_2$$

$$2 = c_1 + c_2$$

$$8 = 4c_1 + 4c_2$$

$$10 = -4c_1 + 2c_2 \Rightarrow \underline{\underline{10 = -4c_1 + 2c_2}}$$

$$18 = 6c_2 \Rightarrow c_2 = 3 \\ c_1 = -1$$

$$y(x) = 3e^{2x} - e^{-4x}$$

2. (12 points) Consider the equation $xy'' + 5y' = 0$, $0 < x < \infty$.

(a) Verify that $y_1(x) = 1$ and $y_2(x) = \frac{1}{x^4}$ are solutions.

$$\begin{aligned} y_1(x) &= 1 \\ y_1' &= 0, \quad y_1'' = 0 \\ x(0) + 5(0) &= 0 \quad \checkmark \end{aligned}$$

$$\begin{aligned} y_2(x) &= x^{-4} \\ y_2' &= -4x^{-5} \\ y_2'' &= 20x^{-6} \\ x(20x^{-6}) + 5(-4x^{-5}) &= 20x^{-5} - 20x^{-5} = 0 \quad \checkmark \end{aligned}$$

Both are solutions.

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(0, \infty)$.

$$W[y_1, y_2](x) = \begin{vmatrix} 1 & x^{-4} \\ 0 & -4x^{-5} \end{vmatrix} = -4x^{-5} = \frac{-4}{x^5} \neq 0$$

(c) Use what you've learned in parts (a) and (b) to find the solution of the IVP
 $xy'' + 5y' = 0$; $y(1) = 2$, $y'(1) = 2$.

$$\begin{aligned} y(x) &= c_1 + \frac{c_2}{x^4} & y(1) &= 2 \Rightarrow 2 = c_1 + c_2 & c_2 &= -\frac{1}{2} \\ y'(x) &= -\frac{4c_2}{x^5} & y'(1) &= 2 \Rightarrow 2 = -4c_2 & c_1 &= \frac{5}{2} \end{aligned}$$

$$y(x) = \frac{5}{2} - \frac{1}{2x^4}$$

(d) Is your solution in part (c) unique? Explain.

$$\text{STANDARD FORM: } y'' + \frac{5}{x}y' = 0$$

$$\left. \begin{array}{l} p(x) = \frac{5}{x} \\ q(x) = 0 \end{array} \right\} \text{CONTINUOUS FUNCTIONS ON } (0, \infty) \Rightarrow \begin{array}{l} \text{By our EXISTENCE/} \\ \text{UNIQUENESS THM,} \end{array}$$

THE SOL'N IS

UNIQUE.

3. (10 points) Use reduction of order to solve: $xy'' + y' = 4x$

$$u = y', \quad u' = y'' \Rightarrow \quad xu' + u = 4x$$

$$u' + \frac{1}{x}u = 4$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x| = x, \quad x > 0$$

$$x u(x) = \int 4x dx = 2x^2 + C$$

$$u(x) = 2x + \frac{C}{x}$$

$$y'(x) = 2x + \frac{C}{x}$$

$$\Rightarrow \boxed{y(x) = x^2 + C \ln|x| + D}$$

4. (10 points) Solve: $(\underbrace{xy - 1}_M dx + \underbrace{x^2 - xy}_N dy = 0)$

$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 2x - y$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{x(x-y)} = \frac{-x+y}{x(x-y)} = -\frac{1}{x}$$

$$\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{x}, \quad x > 0$$

$$(y - \frac{1}{x}) dx + (x - y) dy = 0$$

$$\frac{\partial F}{\partial x} = y - \frac{1}{x} \Rightarrow F(x,y) = xy - \ln|x| + g(y)$$

$$\frac{\partial F}{\partial y} = x - y \Rightarrow F(x,y) = xy - \frac{1}{2}y^2 + h(x)$$

$xy - \ln|x| - \frac{1}{2}y^2 = C$

5. (8 points) Solve: $y'' - 6y' + 9y = 0$

Char eq is $r^2 - 6r + 9 = 0$

$$(r-3)^2 = 0$$

Fund sol'n set is $\{e^{3x}, xe^{3x}\}$

$$y(x) = c_1 e^{3x} + c_2 x e^{3x}$$

Show all work. Supply explanations when necessary. YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. (12 points) Solve: $2yy'' = 1 + (y')^2$; $y(0) = 1$, $y'(0) = 0$

Let $u = y'$. Then $y'' = u \frac{du}{dy}$ AND $u(1) = 0$.

$$2yu \frac{du}{dy} = 1 + u^2$$

$$\frac{2u}{1+u^2} du = \frac{1}{y} dy$$

$$\int \frac{2u}{1+u^2} du = \int \frac{1}{y} dy$$

$$w = 1+u^2$$

$$dw = 2u du$$

$$\int \frac{1}{w} dw$$

$$\ln|w| = \ln|y| + C_1$$

$$\ln(1+u^2) = \ln|y| + C_1$$

$$1+u^2 = C_2|y|$$

$$1+u^2 = C_3 y$$

$$u(1) = 0 \Rightarrow 1 + 0^2 = C_3(1)$$

$$C_3 = 1$$

$$1+u^2 = y$$

Assuming $u \geq 0$,

$$u = \sqrt{y-1}$$

$$\frac{dy}{dx} = \sqrt{y-1}$$

$$\frac{dy}{\sqrt{y-1}} = dx$$

$$\int \frac{1}{\sqrt{y-1}} dy = \int dx$$

$$2(y-1)^{\frac{1}{2}} = x + D_1$$

$$y(0) = 1 \Rightarrow 2(1-1)^{\frac{1}{2}} = 0 + D_1 \\ \Rightarrow D_1 = 0$$

$$2\sqrt{y-1} = x$$

$$y(x) = \frac{x^2}{4} + 1$$

2. (12 points)

- (a) Find the recursive formula for the Taylor method of order three for the initial value problem $y' = e^{-y}$, $y(0) = 0$.

$$y_{N+1} = y_N + h f(x_N, y_N) + \frac{h^2}{2} f'(x_N, y_N) + \frac{h^3}{6} f''(x_N, y_N)$$

$$f(x, y) = e^{-y}$$

$$f'(x, y) = -e^{-y} \frac{dy}{dx} = -e^{-2y}$$

$$\Rightarrow y_{N+1} = y_N + h e^{-y_N} - \frac{h^2}{2} e^{-2y_N} + \frac{h^3}{3} e^{-3y_N}$$

$$f''(x, y) = 2e^{-2y} \frac{dy}{dx} = 2e^{-3y}$$

OR

$$y_{N+1} = y_N + h e^{-y_N} - \frac{1}{2} (h e^{-y_N})^2 + \frac{1}{3} (h e^{-y_N})^3$$

- (b) Use your method with $h = 0.1$ to approximate $y(0.3)$.

$$y_0 = 0, x_0 = 0$$

$$\text{Let } z = e^{-y_0} \approx 0.8333\ldots$$

$$y_1 = 0 + (0.1)(e^0) - \frac{1}{2}(0.1e^0)^2 + \frac{1}{3}(0.1e^0)^3 \\ = 0.0953333$$

$$y_3 = y_2 + (0.1z) - \frac{1}{2}(0.1z)^2 + \frac{1}{3}(0.1z)^3 \\ = 0.26240985$$

$$x_1 = 0.1$$

$$\text{Let } z = e^{-y_1} \approx 0.90906\ldots$$

$$y_2 = y_1 + (0.1z) - \frac{1}{2}(0.1z)^2 + \frac{1}{3}(0.1z)^3 = 0.1823587$$

$$y(0.3) \approx 0.2624$$

- (c) Solve the IVP from part (a) and find the exact value of $y(0.3)$. What is the percent error in your approximation?

LUA
program
IS
ATTACHED.

$$\frac{dy}{dx} = e^{-y}$$

$$y = \ln(x+1)$$

$$e^y dy = dx$$

$$y(0.3) = \ln(1.3) = 0.26236426$$

$$e^y = x + C$$

$$y(0) = 0 \Rightarrow C = 1$$

$$e^y = x + 1$$

$$\% \text{ error} = \left| \frac{y(0.3) - y_3}{y(0.3)} \right| \times 100\%$$

$$\approx 0.017\%$$

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-- Test 2b, Problem #2b
-- Lua Code

xn = 0.0
yn = 0.0

print( "x = ", xn, " y = ", yn )

for i = 1, 3 do
    z = 0.1 * math.exp( -yn )
    yn = yn + z - z^2 / 2 + z^3 / 3
    xn = xn + 0.1
    print( "x = ", xn, " y = ", yn )
end
```

Output:

x =	0	y =	0
x =	0.1	y =	0.095333333333333
x =	0.2	y =	0.18235870021459
x =	0.3	y =	0.2624098537697

3. (12 points) Solve: $y'' + 10y' + 25y = 0$; $y(0) = 2$, $y'(0) = 4$

Char eq: $r^2 + 10r + 25 = 0$

$$(r+5)^2 = 0 \Rightarrow r = -5, \text{ mult 2}$$

Func sol'n set is $\{e^{-5x}, xe^{-5x}\}$

$$y(x) = c_1 e^{-5x} + c_2 x e^{-5x}$$

$$y(0) = 2 \Rightarrow 2 = c_1$$

$$y'(x) = -5c_1 e^{-5x} + c_2 e^{-5x} - 5c_2 x e^{-5x}$$

$$y'(0) = 4 \Rightarrow 4 = -5c_1 + c_2$$

$$\Rightarrow c_2 = 14$$

$$y(x) = 2e^{-5x} + 14xe^{-5x}$$

4. (10 points) Consider the following initial value problem:

$$\frac{dy}{dx} = \frac{y-3}{x}, \quad y(1) = 5 \quad y_{N+1} = y_N + f(x_N, y_N)$$

(a) Use Euler's method with $h = 1$ to approximate $y(6)$. Do the computations by hand—they should be very easy.

$$y_0 = 5, \quad x_0 = 1$$

$$y_1 = 5 + \frac{5-3}{1} = 7, \quad x_1 = 2$$

$$y_2 = 7 + \frac{7-3}{2} = 9, \quad x_2 = 3$$

$$y_3 = 9 + \frac{9-3}{3} = 11, \quad x_3 = 4$$

$$y_4 = 11 + \frac{11-3}{4} = 13, \quad x_4 = 5$$

$$y_5 = 13 + \frac{13-3}{5} = 15, \quad x_5 = 6$$

$$y(6) \approx 15$$

(b) Solve the initial value problem and find the exact value of $y(6)$.

$$\frac{dy}{y-3} = \frac{dx}{x} \rightarrow y(1) = 5 \Rightarrow 5 = C_a + 3 \\ C_a = 2$$

$$\ln|y-3| = \ln|x| + C,$$

$$y-3 = C_a x$$

$$y = C_a x + 3$$

$$y(x) = 2x + 3$$

$$y(6) = 15$$

(c) Find the percent error in your approximation of $y(6)$? Why do you think Euler's method worked so well? Would it work as well with $h = 10$?

Euler's method produced THE EXACT SOLUTION.

THERE IS NO ERROR.

Euler's method worked so well because THE
SOLUTION IS A LINEAR FUNCTION. Euler's method
IS EXACT, REGARDLESS OF STEP SIZE, ANY TIME THE
SOLUTION IS A LINEAR FUNCTION.

5. (4 points) An object is launched into the air so that its velocity, in meters per second, at any time t (in seconds) satisfies the initial value problem

$$v' = -0.8v - 9.8, \quad v(0) = 50.$$

Recalling that a launched object reaches a maximum height when its velocity is zero, use the fourth-order Runge-Kutta method to approximate when the object will reach its highest point. (If you're using the TI-83 program, the intermediate results are stored in your calculator's lists. To see the lists enter **STAT** 1.)

SEE THE OUTPUT OF THE RK4 PROGRAM
(ATTACHED)

$v(t) = 0$ BETWEEN

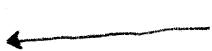
$t = 2.0320$ SECONDS

AND

$t = 2.0321$ SECONDS

Output of RK4 taking 40 steps with $h=0.1$

0, 50
0.1, 45.21399424
0.2, 40.795953960076
0.3, 36.717588639633
0.4, 32.952782835896
0.5, 29.477428956047
0.6, 26.26927288643
0.7, 23.307771490358
0.8, 20.573961061995
0.9, 18.050335893983
1, 15.720736181236
1.1, 13.570244543086
1.2, 11.585090501193
1.3, 9.7525623015333
1.4, 8.0609255158321
1.5, 6.4993479012184
1.6, 5.0578300369377
1.7, 3.7271412939676
1.8, 2.4987607275217
1.9, 1.3648225139509
2, 0.31806558265535
2.1, -0.6482128795237
2.2, -1.5402003491605
2.3, -2.3636085871905
2.4, -3.1237102136535
2.5, -3.8253724704387
2.6, -4.4730883882273
2.7, -5.0710055572065
2.8, -5.6229526857883
2.9, -6.1324641173965
3, -6.6028024623145
3.1, -7.0369794895148
3.2, -7.4377754122486
3.3, -7.8077566908895
3.4, -8.1492924670296
3.5, -8.4645697340637
3.6, -8.7556073414027
3.7, -9.024268921993
3.8, -9.2722748259215
3.9, -9.5012131365213
4, -9.7125498395193



VELOCITY IS ZERO SOMEWHERE

BETWEEN

$t = 2$ AND $t = 2.1$

Part of the data from RK4 with h=0.01

2, 0.31805824585826
2.01, 0.21791488742591
2.02, 0.11856947980202
2.03, 0.020015664846595
2.04, -0.077752864918157
2.05, -0.17474236671151

Part of the data from RK4 with h=0.001

2.0299999999999, 0.020015664161975
2.0309999999999, 0.010203576988826
2.0319999999999, 0.00039933634638453
2.0329999999999, -0.0093970640400633
2.0339999999999, -0.019185630440214
2.0349999999999, -0.028966369118751

Part of the data from RK4 with h=0.0001

2.0316999999999, 0.0033397850267187
2.0317999999999, 0.0023595570535583
2.0318999999999, 0.0013794074954991
2.0319999999999, 0.0003993363462681
2.0320999999999, -0.00058065640040707
2.0321999999999, -0.0015605707507984
2.0322999999999, -0.0025404067111773