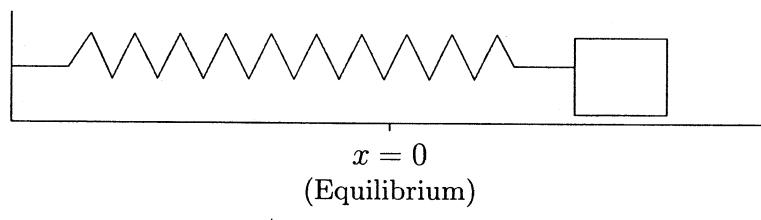


Math 216 - Test 3a
November 22, 2010

Name key _____
Score _____

Show all work. Supply explanations when necessary.

1. (12 points) A 9-kg mass is attached to a spring with spring constant 2 N/m. The mass is moved 1 m to the left of equilibrium (compressing the spring) and released from rest. Assume that there are no damping forces acting on the mass-spring system. Set up and solve the initial value problem that describes the equation of motion of the mass. What is the period of the oscillations?



$$9x'' + 2x = 0; \quad x(0) = -1, \quad x'(0) = 0$$

$$9r^2 + 2 = 0$$

$$\Rightarrow r = \pm \frac{\sqrt{2}}{3} i \Rightarrow \alpha = 0, \beta = \frac{\sqrt{2}}{3}$$

$$x(t) = c_1 \cos \frac{\sqrt{2}}{3} t + c_2 \sin \frac{\sqrt{2}}{3} t$$

$$x(0) = -1 \Rightarrow -1 = c_1$$

$$x'(t) = -\frac{\sqrt{2}}{3} c_1 \sin \frac{\sqrt{2}}{3} t + \frac{\sqrt{2}}{3} c_2 \cos \frac{\sqrt{2}}{3} t$$

$$x'(0) = 0 \Rightarrow 0 = \frac{\sqrt{2}}{3} c_2 \Rightarrow c_2 = 0$$

$$x(t) = -\cos \frac{\sqrt{2}}{3} t$$

or $x(t) = \sin \left(\frac{\sqrt{2}}{3} t - \frac{\pi}{2} \right)$

$$\omega = \frac{\sqrt{2}}{3}, \quad f = \frac{\sqrt{2}}{6\pi}$$

$$P_{\text{period}} = T = \frac{6\pi}{\sqrt{2}} = 3\sqrt{2}\pi \approx [13.33 \text{ sec}]$$

2. (8 points) Find the general solution of the following equation.

$$y'' - 4y' + 7y = 0$$

$$r^2 - 4r + 7 = 0$$

COMPLETE SQUARE ...

$$r^2 - 4r + 4 = -3$$

$$(r-2)^2 = -3 \Rightarrow r-2 = \pm \sqrt{3} i$$

$$r = 2 \pm \sqrt{3} i$$

$$\alpha = 2, \beta = \sqrt{3}$$

$$y(x) = c_1 e^{2x} \cos \sqrt{3} x$$

$$+ c_2 e^{2x} \sin \sqrt{3} x$$

3. (6 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a) $x(t) = 2e^{-2t} + 5te^{-2t} \leftarrow$ REPEATED ROOT $\Rightarrow b^2 - 4mk = 0 \Rightarrow$ CRIT DAMPED

(b) $x'' + 8x' + 17x = 0 \leftarrow b^2 - 4mk = 64 - 4(17) = -4 \Rightarrow$ UNDERDAMPED

(c) $x(t) = \sqrt{6} \sin(4t + \pi) \leftarrow$ NO EXPONENTIAL \Rightarrow SIMPLE HARMONIC

(d) $2x'' + 5x' + 3x = 0 \leftarrow b^2 - 4mk = 25 - 4(2)(3) = 1 \Rightarrow$ OVERDAMPED

4. (8 points) Consider the following equation:

$$y'' - 12y' + 36y = (3x + 5)e^{6x}.$$

Solve the corresponding homogeneous equation. Then use your undetermined coefficients table to find the appropriate form of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

Homo eq: $y'' - 12y' + 36y = 0$

$$r^2 - 12r + 36 = 0$$

$$(r-6)^2 = 0 \Rightarrow r = 6$$

$$\boxed{y_h(x) = C_1 e^{6x} + C_2 x e^{6x}}$$

Non Homo :

$$y'' - 12y' + 36y = (3x + 5)e^{6x}$$

$$g(x) = (3x + 5)e^{6x}$$

$$y_p(x) = x^s (Ax + B)e^{6x}$$

$s = 0$: No, HAS terms $x e^{6x} \notin e^{6x}$

$s = 1$: No, HAS term $x e^{6x}$

$s = 2$: Yes!

$$\boxed{y_p(x) = (Ax^3 + Bx^2)e^{6x}}$$

5. (16 points) Find the general solution of the following equation.

$$2y'' + 6y' - 10y = 260 \cos x$$

$$y'' + 3y' - 10y = 130 \cos x$$

Homo eq: $y'' + 3y' - 10y = 0$

$$r^2 + 3r - 10 = 0$$

$$(r+5)(r-2) = 0$$

$$r = -5, r = 2$$

$$y_h(x) = c_1 e^{-5x} + c_2 e^{2x}$$

Cramer's rule ...

$$A = \frac{\begin{vmatrix} 130 & 3 \\ 0 & -11 \end{vmatrix}}{\begin{vmatrix} -11 & 3 \\ -3 & -11 \end{vmatrix}} = \frac{130(-11)}{130} = -11$$

$$B = \frac{\begin{vmatrix} -11 & 130 \\ -3 & 0 \end{vmatrix}}{\begin{vmatrix} -11 & 3 \\ -3 & -11 \end{vmatrix}} = \frac{3(130)}{130} = 3$$

NonHomo eq: $y'' + 3y' - 10y = 130 \cos x$

$$g(x) = 130 \cos x$$

$$y_p(x) = x^s (A \cos x + B \sin x)$$

$$\text{CHOOSE } s = 0$$

$$y_p(x) = A \cos x + B \sin x$$

$$y'_p(x) = -A \sin x + B \cos x$$

$$y''_p(x) = -A \cos x - B \sin x$$

$$y_p(x) = -11 \cos x + 3 \sin x$$

$$y(x) = c_1 e^{-5x} + c_2 e^{2x}$$

$$-11 \cos x + 3 \sin x$$

SUBS INTO EQUATION ...

$$(-A \cos x - B \sin x) + 3(-A \sin x + B \cos x) - 10(A \cos x + B \sin x) = 130 \cos x$$

$$-A + 3B - 10A = 130$$

$$-11A + 3B = 130$$

$$-B - 3A - 10B = 0$$

$$\Rightarrow -3A - 11B = 0$$

Math 216 - Test 3b
 November 22, 2010

Name key Score _____

Show all work. Supply explanations when necessary.

YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. (11 points) One solution of the equation

$$xy'' - (2x+1)y' + (x+1)y = 0$$

is $y_1(x) = e^x$. Find the general solution.

$$y'' - \frac{2x+1}{x}y' + \frac{x+1}{x}y = 0$$

$$-\int p(x)dx = \int \frac{2x+1}{x}dx = \int \left(2 + \frac{1}{x}\right)dx = 2x + \ln|x| \\ = 2x + \ln x, \quad x > 0$$

$$e^{-\int p(x)dx} = e^{2x + \ln x} = x e^{2x}$$

$$V(x) = \int \frac{x e^{2x}}{(e^x)^a} dx = \int x dx = \frac{1}{2}x^2$$

$$y_2(x) = V(x) y_1(x) = \frac{1}{2}x^2 e^x$$

$$y(x) = c_1 e^x + c_2 x^2 e^x$$

STANDARD FORM FOR VARIATION OF PARAMETERS : $y'' + \frac{5}{x}y' + \frac{4}{x^2}y = 5x$

2. (15 points) For $x > 0$, consider the following nonhomogeneous Cauchy-Euler equation:

$$x^2y'' + 5xy' + 4y = 5x^3.$$

(a) Solve the corresponding homogeneous equation. $x^2y'' + 5xy' + 4y = 0$

LET $x = e^t$. THIS TRANSFORMS THE EQUATION TO $\frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 4y = 0$

CHAR eq is $r^2 + 4r + 4 = 0$

$$(r+2)^2 = 0$$

$$r = -2, \text{ mult 2}$$

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$\rightarrow y(x) = c_1 x^{-2} + c_2 (\ln x) x^{-2}$$

$$\text{OR } y_h(x) = \frac{c_1}{x^2} + \frac{c_2 \ln x}{x^2}$$

(b) Use variation of parameters to solve the nonhomogeneous equation. Evaluate all integrals by hand.

$$y_1(x) = x^{-2}, \quad y_2(x) = x^{-2} \ln x$$

$$g(x) = 5x$$

$$W[y_1, y_2](x) = \begin{vmatrix} x^{-2} & x^{-2} \ln x \\ -2x^{-3} & x^{-3} - 2x^{-3} \ln x \end{vmatrix}$$

$$= x^{-5} = \frac{1}{x^5}$$

$$v_1(x) = \int -5x x^{-2} \ln x x^5 dx$$

$$= \int -5x^4 \ln x dx = -x^5 \ln x + \int x^4 dx$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$dv = -5x^4 dx \quad v = -x^5$$

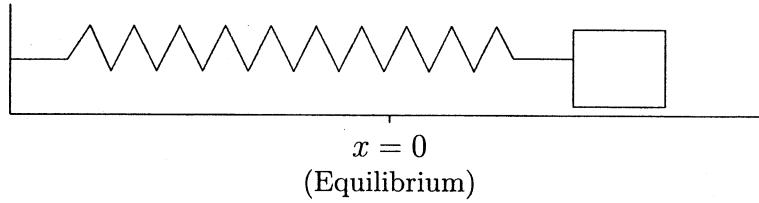
$$v_1(x) = -x^5 \ln x + \frac{1}{5} x^5$$

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$= -x^3 \ln x + \frac{1}{5} x^3 + x^3 \ln x = \frac{1}{5} x^3$$

$$y(x) = \frac{c_1}{x^2} + \frac{c_2 \ln x}{x^2} + \frac{1}{5} x^3$$

3. (13 points) A 1/4-kg mass is attached to a spring with spring constant 8 N/m. The damping constant for the system is 1/4 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and released from rest. Find the equation of motion. Write your final result in terms of a single trig function with phase shift. Graph your solution¹ and attach a copy.



$$\frac{1}{4}x'' + \frac{1}{4}x' + 8x = 0; \quad x(0) = 1, \quad x'(0) = 0$$

$$x'' + x' + 32x = 0$$

$$r^2 + r + 32 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1 - 128}}{2} = -\frac{1}{2} \pm \frac{\sqrt{127}}{2} i$$

$$\alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{127}}{2}$$

$$x(t) = c_1 e^{-t/2} \cos \frac{\sqrt{127}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{127}}{2} t$$

$$x'(t) = c_1 \left(-\frac{1}{2}\right) e^{-t/2} \cos \frac{\sqrt{127}}{2} t - c_1 \left(\frac{\sqrt{127}}{2}\right) e^{-t/2} \sin \frac{\sqrt{127}}{2} t + c_2 \left(-\frac{1}{2}\right) e^{-t/2} \sin \frac{\sqrt{127}}{2} t + c_2 \left(\frac{\sqrt{127}}{2}\right) e^{-t/2} \cos \frac{\sqrt{127}}{2} t$$

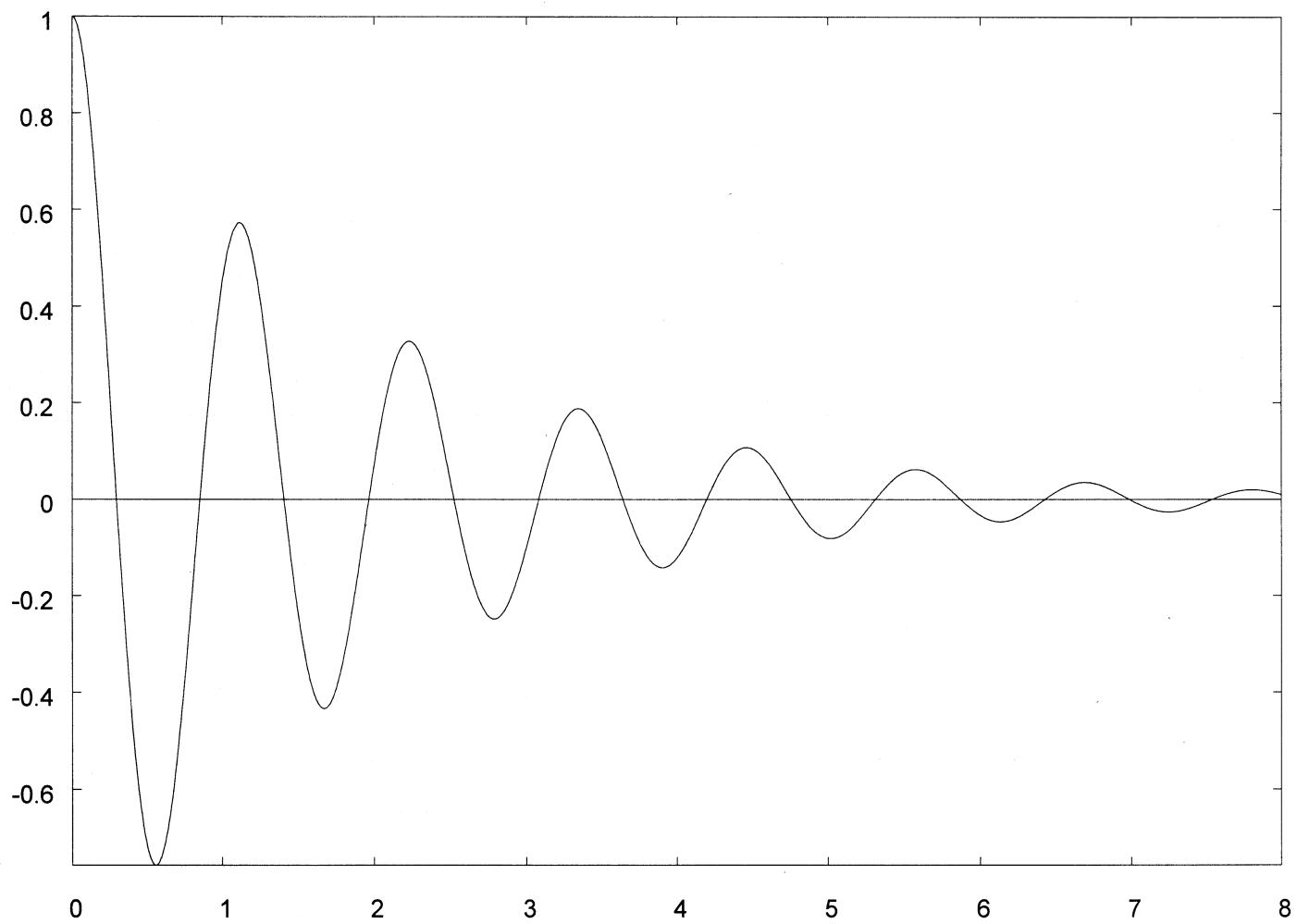
$$x(0) = 1 \Rightarrow c_1 = 1 = A \sin \varphi$$

$$x'(0) = 0 \Rightarrow -\frac{1}{2}c_1 + c_2 \frac{\sqrt{127}}{2} = 0 \Rightarrow c_2 = \frac{1}{\sqrt{127}} = A \cos \varphi$$

$$A = \sqrt{1 + \frac{1}{127}} = \sqrt{\frac{128}{127}} \quad \tan \varphi = \sqrt{127} \quad \varphi \text{ IS IN QUAO I} \\ \text{SINCE } c_1, c_2 > 0.$$

$$x(t) = \sqrt{\frac{128}{127}} e^{-t/2} \sin \left(\frac{\sqrt{127}}{2} t + \tan^{-1} \sqrt{127} \right)$$

¹If you don't have a good plotting program, try one that is available online such as <http://fooplot.com>.



4. (11 points) Find the general solution of the following equation.

$$y^{(5)} + y''' - 2y' = 0$$

Char eq is $r^5 + r^3 - 2r = 0$

$$r(r^4 + r^2 - 2) = 0$$

$$r(r^2 - 1)(r^2 + 2) = 0$$

$$r(r+1)(r-1)(r^2 + 2) = 0$$

$$r=0, r=-1, r=1, \underbrace{r = \pm \sqrt{2}i}_{\alpha=0, \beta=\sqrt{2}} \Rightarrow \left\{ e^{0x}, e^{-x}, e^x, e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x \right\}$$

$$y(x) = c_1 + c_2 e^{-x} + c_3 e^x + c_4 \cos \sqrt{2}x + c_5 \sin \sqrt{2}x$$