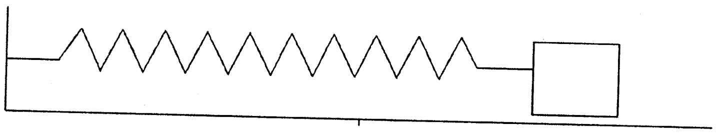
Math 216 - 2nd Final Exam
December 13, 2010

Name key
Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) A 3-kg mass is attached to a spring with spring constant 3 N/m. The damping constant for the system is 6 N-sec/m. The mass is moved 1 m to the LEFT of equilibrium (compressing the spring) and pushed to the RIGHT at 1 m/sec. Set up and solve the initial value problem that describes the displacement of the mass from equilibrium. Is the mass-spring system underdamped, overdamped, or critically damped?



$x = 0$
(Equilibrium)

$3x'' + 6x' + 3x = 0, \quad x(0) = -1, \quad x'(0) = 1$

$3r^2 + 6r + 3 = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \quad \Longrightarrow$ System is critically damped

$r = -1$

$x(t) = c_1 e^{-t} + c_2 t e^{-t}$

$x(0) = -1 \Rightarrow c_1 = -1$

$x'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$x'(0) = -c_1 + c_2 = 1$

$\Rightarrow 1 + c_2 = 1$

$\Rightarrow c_2 = 0$

$\boxed{x(t) = -e^{-t}}$

2. (10 points) Find the general solution of $y''' + 2y'' + 2y' = 0$.

$$r^3 + 2r^2 + 2r = 0$$

$$r(r^2 + 2r + 2) = 0$$

$r = 0$ $\quad (r+1)^2 = -1$

$r = -1 \pm i$

$$\{e^{0x}, e^{-x}\sin x, e^{-x}\cos x\}$$

$$\boxed{y(x) = c_1 + c_2 e^{-x}\sin x + c_3 e^{-x}\cos x}$$

3. (15 points) Solve: $\quad xy' + 2y = x^2, \quad y(1) = 1$

$$y' + \frac{2}{x}y = x$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = |x|^2 = x^2$$

$$x^2 y(x) = \int x^2 x \, dx$$

$$x^2 y(x) = \frac{1}{4}x^4 + C$$

$$y(1) = 1 \Rightarrow 1 = \frac{1}{4} + C$$

$$\Rightarrow C = \frac{3}{4}$$

$$x^2 y(x) = \frac{1}{4}x^4 + \frac{3}{4}$$

$$\boxed{y(x) = \frac{1}{4}x^2 + \frac{3}{4x^2}}$$

4. (15 points) One solution of the following equation is $y(x) = \sqrt{x}$.

$$4x^2y'' - (20x^2 + 4x)y' + (10x + 3)y = 0$$

Find the general solution. (Hint: Do not forget to rewrite the equation in standard form.)

$$y'' - \left(5 + \frac{1}{x}\right)y' + \left(\frac{10x+3}{4x^2}\right)y = 0$$

$$\int p(x)\,dx = -\int \left(5 + \frac{1}{x}\right)dx = -5x - \ln x, \quad x > 0$$

$$e^{-\int p(x)\,dx} = e^{5x + \ln x} = x e^{5x}$$

$$\boxed{y(x) = c_1\sqrt{x} + c_2\sqrt{x}\,e^{5x}}$$

$$V(x) = \int \frac{x e^{5x}}{x}\,dx = \frac{1}{5}e^{5x}$$

$$\Rightarrow \quad y_2(x) = \frac{1}{5}\sqrt{x}\,e^{5x}$$

5. (12 points) Solve: $\quad y' = xy^{-3}e^x, \quad y(0) = 1$

$$y^3\,dy = xe^x\,dx$$

$$\int y^3\,dy = \int xe^x\,dx$$

$u = x \qquad du = dx$

$dv = e^x\,dx \qquad v = e^x$

$$\frac{y^4}{4} = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

$y(0) = 1$

$$\Rightarrow \frac{1}{4} = -1 + C$$

$$\Rightarrow C = \frac{5}{4}$$

$$\frac{y^4}{4} = xe^x - e^x + \frac{5}{4}$$

$$\boxed{y(x) = \sqrt[4]{4xe^x - 4e^x + 5}}$$

6. (12 points) Show that the following equation is exact and solve.

$$\underbrace{(4xy^{1/2} + x^2 - 2x^{-1/2}y^{1/2})}_{M}\, dx + \underbrace{(2y + x^2y^{-1/2} - 2x^{1/2}y^{-1/2})}_{N}\, dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2xy^{-1/2} - x^{-1/2}y^{-1/2} \\ \frac{\partial N}{\partial x} &= 2xy^{-1/2} - x^{-1/2}y^{-1/2} \end{aligned}\right\} \text{EQUAL} \Rightarrow \text{EXACT!}$$

$$\frac{\partial F}{\partial x} = 4xy^{1/2} + x^2 - 2x^{-1/2}y^{1/2} \Rightarrow F(x,y) = 2x^2y^{1/2} + \frac{1}{3}x^3 - 4x^{1/2}y^{1/2} + g(y)$$

$$\frac{\partial F}{\partial y} = 2y + x^2y^{-1/2} - 2x^{1/2}y^{-1/2} \Rightarrow F(x,y) = y^2 + 2x^2y^{1/2} - 4x^{1/2}y^{1/2} + h(x)$$

SOL'N : $2x^2y^{1/2} + y^2 + \frac{1}{3}x^3 - 4x^{1/2}y^{1/2} = C$

7. (10 points) Use Euler's method with a step size of $h = 0.5$ to approximate $y(3)$, where $y(x)$ is the solution of the initial value problem $y' = xy^2$, $y(2) = 1$.

$f(x,y) = xy^2$

$x_0 = 2, \quad y_0 = 1$

$y_1 = y_0 + 0.5\, f(x_0, y_0)$

$= 1 + 0.5\,(2 \cdot 1^2) = 2$

$x_1 = 2.5, \quad y_1 = 2$

$y_2 = y_1 + 0.5\, f(x_1, y_1)$

$= 2 + 0.5\,(2.5 \cdot 2^2) = 7$

$x_2 = 3, \quad y_2 = 7$

$\boxed{y(3) \approx 7}$

8. (20 points) Use Laplace transforms to solve the following IVP:

$$y'' + 2y' + 5y = 2e^{-t}; \quad y(0) = 0,\ y'(0) = 1$$

You may use the TI-92 to do any required partial fraction decompositions.

LET $Y(s) = \mathcal{L}\{y(t)\}(s)$.

TRANSFORM BOTH SIDES...

$$s^2 Y(s) - s(0) - 1 + 2\left[sY(s) - 0\right] + 5Y(s) = \frac{2}{s+1}$$

$$(s^2 + 2s + 5)\, Y(s) - 1 = \frac{2}{s+1}$$

$$Y(s) = \frac{\frac{2}{s+1} + 1}{s^2 + 2s + 5} = -\frac{1}{2}\,\frac{s-1}{s^2+2s+5} + \frac{1}{2}\,\frac{1}{s+1}$$

$$= -\frac{1}{2}\left[\frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}\right] + \frac{1}{2}\,\frac{1}{s+1}$$

INVERSE...

$$\boxed{y(t) = -\frac{1}{2}e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + \frac{1}{2}e^{-t}}$$

9. (20 points) Use undetermined coefficients to find the general solution of the following equation:

$$x'' - 4x' + 4x = 2 + 3e^t$$

Homo eq: $x'' - 4x' + 4x = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$, mult 2

$x_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

Non Homo eq:

$g(x) = 2 + 3e^t$

$x_p(t) = A + Be^t$

$x_p'(t) = Be^t$

$x_p''(t) = Be^t$

$Be^t - 4Be^t + 4A + 4Be^t = 2 + 3e^t$

$Be^t + 4A = 2 + 3e^t$

$A = \frac{1}{2}, \quad B = 3$

$x_p(t) = \frac{1}{2} + 3e^t$

$x(t) = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{2} + 3e^t$

10. (10 points) Find the orthogonal trajectories for the family of curves described by the equation $Cy^2 = x^3$.

$2Cy \frac{dy}{dx} = 3x^2$

$2Cy^2 \frac{dy}{dx} = 3x^2 y$

$2x^3 \frac{dy}{dx} = 3x^2 y$

$\frac{dy}{dx} = \frac{3y}{2x}$

Ortho Trajs satisfy

$\frac{dy}{dx} = -\frac{2x}{3y} \Rightarrow 3y\,dy = -2x\,dx$

$\frac{3y^2}{2} = -x^2 + C$

$x^2 + \frac{3y^2}{2} = C$

11. (6 points) For $x > 0$, let $y_1(x) = \ln x^5$ and $y_2(x) = \ln x$. Compute the Wronskian of y_1 and y_2. Briefly explain why $y(x) = c_1 y_1(x) + c_2 y_2(x)$ cannot be the general solution of a 2nd-order, linear, homogeneous differential equation.

$$W[y_1, y_2](x) = \begin{vmatrix} \ln x^5 & \ln x \\ \frac{5}{x} & \frac{1}{x} \end{vmatrix} = \frac{\ln x^5}{x} - \frac{5 \ln x}{x} = 0$$

SINCE $\ln x^5 = 5 \ln x$

y_1 AND y_2 ARE NOT LINEARLY INDEPENDENT !

12. (4 points) What does it mean for two families of curves to be orthogonal trajectories of one another?

AT EACH POINT WHERE A MEMBER OF ONE FAMILY INTERSECTS A MEMBER OF THE OTHER, THE TANGENT LINES ARE PERPENDICULAR.

13. (4 points) Use a substitution to convert the following equation to a first order equation. DO NOT SOLVE. How many constants of integration would your solution have?

$$y^2 y'' - 4y = (y')^2$$

$$u = y', \quad y'' = u \frac{du}{dy}$$

$$y^2 u \frac{du}{dy} - 4y = u^2$$

FINAL SOLUTION IS A $\underline{2}$ PARAMETER FAMILY OF CURVES.