# Math 216 - Quiz 10

November 25, 2015

Name key

Score

Show all work to receive full credit. Supply explanations when necessary.

1. (5 points) Solve the following initial value problem.

$$\begin{aligned} y' - 2y + z &= 0, \qquad & y(0) &= 1 \\ z' - y - 2z &= 0, \qquad & z(0) &= 0 \end{aligned}$$

$$(D-2)y + z = 0$$
$$-y + (D-2)z = 0$$

EXPECT 2 CONSTANTS.

$$(D-2)y + z = 0$$
$$-(D-2)y + (D^2-4D+4)z = 0$$

$$(D^2-4D+5)z = 0$$

$$r^2 - 4r + 5 = 0$$
$$r^2 - 4r + 4 = -1$$
$$(r-2)^2 = -1$$
$$r = 2 \pm i$$

$$z(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$$

$$\begin{aligned} y = z' - 2z = {} & \cancel{2c_1 e^{2t}\cos t} - c_1 e^{2t}\sin t \\ & + \cancel{2c_2 e^{2t}\sin t} + c_2 e^{2t}\cos t \\ & - \cancel{2c_1 e^{2t}\cos t} - \cancel{2c_2 e^{2t}\sin t} \\ = {} & c_2 e^{2t}\cos t - c_1 e^{2t}\sin t \end{aligned}$$

$$y(0) = 1 \Rightarrow c_2 = 1$$
$$z(0) = 0 \Rightarrow c_1 = 0$$

$$\boxed{\begin{aligned} z(t) &= e^{2t}\sin t \\ y(t) &= e^{2t}\cos t \end{aligned}}$$

2. (5 points) Solve the following initial value problem.

$$\begin{aligned} x' &= 3x - y - 1, & x(0) &= 0 \\ y' &= x + y + 4e^t, & y(0) &= -2 \end{aligned}$$

$$\begin{aligned} (D-3)x + y &= -1 \\ -x + (D-1)y &= 4e^t \end{aligned} \Rightarrow \begin{aligned} (D-3)x + y &= -1 \\ -(D-3)x + (D^2-4D+3)y &= -8e^t \end{aligned}$$

$$(D^2-4D+4)y = -1-8e^t$$

Expect 2 constants.

Homo eqn...

$$(r-2)^2 = 0 \Rightarrow y_h(t) = c_1e^{2t} + c_2te^{2t}$$

Non Homo eqn...

$$g(t) = -1-8e^t \Rightarrow y_p(t) = A + Be^t$$

$$y'' - 4y' + 4y = -1-8e^t$$

$$\Downarrow$$

$$Be^t - \cancel{4Be^t} + 4A + \cancel{4Be^t} = -1-8e^t$$

$$A = -\tfrac{1}{4}, \quad B = -8$$

$$y(t) = c_1e^{2t} + c_2te^{2t} - \tfrac{1}{4} - 8e^t$$

Sub into 2nd eqn...

$$\begin{aligned} x &= y' - y - 4e^t \\ &= \cancel{2c_1e^t} + c_2e^{2t} + 2c_2te^{2t} - \cancel{8e^t} \\ &\quad - c_1e^{2t} - c_2te^{2t} + \tfrac{1}{4} + \cancel{8e^t} - 4e^t \\ &= c_1e^{2t} + c_2e^{2t} + c_2te^{2t} + \tfrac{1}{4} - 4e^t \end{aligned}$$

$$x(t) = c_1e^{2t} + c_2e^{2t} + c_2te^{2t} + \tfrac{1}{4} - 4e^t$$

$$x(0) = 0 \Rightarrow c_1 + c_2 + \tfrac{1}{4} - 4 = 0$$

$$y(0) = -2 \Rightarrow c_1 - \tfrac{1}{4} - 8 = -2$$

$$c_1 = \frac{25}{4}$$

$$c_2 = -\frac{5}{2}$$

$$\boxed{\begin{aligned} x(t) &= \frac{15}{4}e^{2t} - \frac{5}{2}te^{2t} + \frac{1}{4} - 4e^t \\ y(t) &= \frac{25}{4}e^{2t} - \frac{5}{2}te^{2t} - \frac{1}{4} - 8e^t \end{aligned}}$$

See next page for Laplace Transform approach.

#2 LAPLACE TRANSFORM APPROACH ...

$$x' - 3x + y = -1 \qquad x(0) = 0$$

$$-x + y' - y = 4e^t \qquad y(0) = -2$$

TRANSFORM ...

$$sX - 3X + Y = -\frac{1}{s}$$
$$-X + sY + 2 - Y = \frac{4}{s-1}$$

$$\Rightarrow$$

$$(s-3)X + Y = -\frac{1}{s}$$
$$-X + (s-1)Y = \frac{4}{s-1} - 2.$$

$$\Downarrow$$

$$(s-3)X + Y = -\frac{1}{s}$$
$$-(s-3)X + (s^2-4s+3)Y = (s-3)\left(\frac{4}{s-1} - 2\right)$$

$$(s^2-4s+4)Y = \frac{(s-3)(-2s+6)}{s-1} - \frac{1}{s}$$

$$Y = \frac{1}{s^2-4s+4}\left(\frac{(s-3)(-2s+6)}{s-1} - \frac{1}{s}\right)$$

PARTIAL FRACTION DECOMP.

$$Y = -\frac{1}{4s} - \frac{8}{s-1} + \frac{25}{4(s-2)} - \frac{5}{2(s-2)^2}$$

SUBS y INTO SECOND EQN.

$$y(t) = -\frac{1}{4} - 8e^t + \frac{25}{4}e^{2t} - \frac{5}{2}te^{2t}$$

$$x(t) = \frac{1}{4} - 4e^t + \frac{15}{4}e^{2t} - \frac{5}{2}te^{2t}$$