Math 216 - Quiz 2

September 2, 2015

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary.

1. (3 points) Solve the initial value problem: $\frac{dy}{dx} = xy^3 + 3x^2y^3, \quad y(1) = 1/2.$

$$\frac{dy}{dx} = y^3(x + 3x^2)$$

$$y^{-3}\,dy = (x + 3x^2)\,dx$$

$$-\frac{1}{2}y^{-2} = \frac{1}{2}x^2 + x^3 + C$$

$$\frac{1}{y^2} = -x^2 - 2x^3 + C$$

$$y^2 = \frac{1}{C - x^2 - 2x^3}$$

Assuming $y > 0$

$$y(x) = \frac{1}{\sqrt{C - x^2 - 2x^3}},$$

$$y(1) = \frac{1}{2} \Rightarrow C = 7$$

$$y(x) = \frac{1}{\sqrt{7 - x^2 - 2x^3}}$$

2. (3 points) Solve the initial value problem: $x\frac{dy}{dx} = 1 + x + y, \quad y(1) = 4.$

$$x\frac{dy}{dx} - y = 1 + x$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x} + 1$$

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{|x|} = \frac{1}{x}, \quad x > 0$$

$$y(x) = x\int\left(\frac{1}{x^2} + \frac{1}{x}\right)dx$$

$$= x\left[-\frac{1}{x} + \ln|x| + C\right]$$

$$y(x) = x\ln x - 1 + Cx$$

$$y(1) = 4 \Rightarrow C = 5$$

$$y(x) = x\ln x - 1 + 5x, \quad x > 0$$

3. (4 points) A tank initially contains 10 gal of pure water. A salt solution containing 4 lb of salt per gallon enters the tank at 2 gal/min and is uniformly mixed. The mixed solution leaves the tank at 3 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes ($0 \le t \le 10$). Set up and solve the appropriate initial value problem to determine $A(t)$.

$$\frac{4\text{ lb}}{\text{gal}} \cdot \frac{2\text{ gal}}{\text{min}} = \frac{8\text{ lb}}{\text{min}}$$

$V(0) = 10$, $A(0) = 0$

$V(t) = 10 - t$, $A(t) = ?$

$$\frac{3\text{ gal}}{\text{min}} \cdot \frac{A(t)\text{ lb}}{V(t)\text{ gal}} = \frac{3A(t)}{V(t)}\ \frac{\text{lb}}{\text{min}} = \frac{3A}{10-t}$$

$$\frac{dA}{dt} = 8 - \frac{3A}{10-t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{3}{10-t}A = 8$$

$$\mu(t) = e^{\int \frac{3}{10-t}\,dt} = e^{-3\ln|10-t|} = \frac{1}{(10-t)^3}, \quad 0 \le t \le 10$$

$$A(t) = (10-t)^3 \int \frac{8}{(10-t)^3}\,dt$$

$$= (10-t)^3\left[\frac{4}{(10-t)^2} + C\right]$$

$$= 4(10-t) + C(10-t)^3$$

$A(0) = 0$

$\Rightarrow 40 + 1000C$

$\Rightarrow C = \frac{-4}{100}$

$$A(t) = 4(10-t) - \frac{1}{25}(10-t)^3$$