Math 216 - Quiz 5
September 30, 2015

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary.

1. (3 points) Find the orthogonal trajectories for the family of curves described by $y^2 = cx^3$.

$2y\frac{dy}{dx} = 3cx^2$

$2y\frac{dy}{dx} = 3\left(\frac{y^2}{x^3}\right)x^2$

$\frac{dy}{dx} = \frac{3y}{2x}$

$\frac{dy}{dx} = -\frac{2x}{3y}$

$3y\,dy = -2x\,dx$

$\frac{3}{2}y^2 = -x^2 + C$

$x^2 + \frac{3}{2}y^2 = C$

2. (3 points) An object of mass 8 kg is given an upward initial velocity of 20 m/sec and then allowed to fall under the influence of gravity. Assume that the force in Newtons due to air resistance is $-16v$, where v is the velocity of the object in m/sec. Determine the equation of motion of the object. If the object is initially 100 m above the ground, determine when the object will strike the ground.

$8v' + 16v = -8(9.8) \quad v(0) = 20$

$v' + 2v = -9.8, \quad v(0) = 20$

$\mu(t) = e^{\int 2\,dt} = e^{2t}$

$v(t) = e^{-2t}\int -9.8e^{2t}\,dt$

$= e^{-2t}\left(-4.9e^{2t} + C\right)$

$= -4.9 + Ce^{-2t}$

$v(0) = 20 \Rightarrow C = 24.9$

$v(t) = -4.9 + 24.9e^{-2t}$

$x(t) = -4.9t - 12.45e^{-2t} + C$

$x(0) = 100 \Rightarrow C = 112.45$

$x(t) = -4.9t - 12.45e^{-2t} + 112.45$

$x(t) = 0$ about when

$-4.9t + 112.45 = 0$

$t \approx 22.9$ sec

3. (4 points) The US population from 1790 to 1940 can be approximated by the solution of the initial value problem

$$\frac{dP}{dt} = 0.0318P - 0.000170P^2, \quad P(0) = 3.9,$$

where P is in millions and t is in years since 1790.

(a) Solve for $P(t)$.

$$\frac{dP}{dt} - 0.0318P = -0.00017P^2$$

$$P^{-2}\frac{dP}{dt} - 0.0318P^{-1} = -0.00017$$

$$u = P^{-1}$$

$$\frac{du}{dt} = -P^{-2}\frac{dP}{dt}$$

$$\frac{du}{dt} + 0.0318u = 0.00017$$

$$\mu(t) = e^{\int 0.0318\,dt} = e^{0.0318t}$$

$$u(t) = e^{-0.0318t}\int 0.00017e^{0.0318t}\,dt$$

$$u(t) = e^{-0.0318t}\left(\frac{0.00017}{0.0318}e^{0.0318t} + C\right)$$

$$\frac{1}{P(t)} = \frac{0.00017}{0.0318} + Ce^{-0.0318t}$$

$$P(0) = 3.9 \Rightarrow C = \frac{1}{3.9} - \frac{0.00017}{0.0318}$$

$$= 0.25106$$

$$P(t) = \frac{1}{0.005346 + 0.25106e^{-0.0318t}}$$

(b) If the actual population in 1900 was 76.0 million, find the percent error in the approximation given by P.

$$P(110) = 77.26 \text{ million}$$

$$\frac{|77.26 - 76|}{76} \times 100\% \approx 1.7\%$$