Math 216 - Test 1

September 16, 2015

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2} = x^2 + y^2$ PARTIAL, LINEAR, 2ND ORDER

   (b) $(x^2 + y^2)\,dx + 2xy\,dy = 0$ ORDINARY, NONLINEAR, 1ST ORDER

   (c) $t^2x'' - 3tx' + 2x = t^2 \sin 5t$ ORDINARY, LINEAR, 2ND ORDER

   (d) $y\dfrac{dy}{dx} - e^x y = -3x + 5$ ORDINARY, NONLINEAR, 1ST ORDER

2. (12 points) Solve the following initial value problem:

$$y\,dy = 4x(y^2+1)^{1/2}\,dx, \quad y(0) = 1$$

$$\frac{y}{(y^2+1)^{1/2}}\,dy = 4x\,dx$$

$$\int \frac{y}{(y^2+1)^{1/2}}\,dy = \int 4x\,dx$$

$u = y^2+1$

$du = 2y\,dy$

$$\frac{1}{2}\int u^{-1/2}\,du = \int 4x\,dx$$

$$u^{1/2} = 2x^2 + C$$

$$(y^2+1)^{1/2} = 2x^2 + C$$

$$y^2 + 1 = (2x^2+C)^2$$

$y(x) = \sqrt{(2x^2+C)^2 - 1}$, $y > 0$

$y(0) = 1 \Rightarrow \sqrt{C^2 - 1} = 1 \Rightarrow C = \sqrt{2}$

$$y(x) = \sqrt{(2x^2+\sqrt{2})^2 - 1}$$

3. (14 points) Consider the following initial value problem:

$$(1 - 2x^2 - 2y)\,dy - (4x^3 + 4xy)\,dx = 0, \quad y(2) = 1.$$

(a) Use the test for exactness to show that the DE is exact?

$M = -4x^3 - 4xy \qquad N = 1 - 2x^2 - 2y$

$\frac{\partial M}{\partial y} = -4x \quad = \quad \frac{\partial N}{\partial x} = -4x$

(b) Solve the initial value problem.

$\frac{\partial F}{\partial x} = -4x^3 - 4xy \Rightarrow F(x,y) = -x^4 - 2x^2y + g(y)$

$\frac{\partial F}{\partial y} = 1 - 2x^2 - 2y \Rightarrow F(x,y) = y - 2x^2y - y^2 + h(x)$

$F(x,y) = -x^4 - 2x^2y + y - y^2$

$F(2,1) = -16 - 8 + 1 - 1 = -24$

SOL'N IS

$-x^4 - 2x^2y + y - y^2 = -24$

(c) Is your solution explicit or implicit?

$\boxed{\text{IMPLICIT}}$

4. (12 points) Solve: $\dfrac{dy}{dx} = \dfrac{-xy^4}{(y^2+2)e^{-3x}}$

$$\frac{y^2+2}{y^4}\,dy = -xe^{3x}\,dx$$

$$\int \left(y^{-2}+2y^{-4}\right)dy = -\int xe^{3x}\,dx$$

+	x	e^{3x}
−	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$$-\frac{1}{y} - \frac{2}{3y^3} = -\frac{1}{3}xe^{3x} + \frac{1}{9}e^{3x} + C$$

$$\boxed{-\frac{1}{y^3}\left(y^2+\frac{2}{3}\right) = e^{3x}\left(\frac{1}{9}-\frac{x}{3}\right) + C}$$

5. (12 points) Solve the following initial value problem:

$$x\frac{dy}{dx} + (3x+1)y = e^{-3x}, \quad y(1) = 0$$

$$\frac{dy}{dx} + \frac{3x+1}{x}y = \frac{1}{x}e^{-3x}$$

$$\mu(x) = e^{\int 3+\frac{1}{x}\,dx} = e^{3x+\ln|x|} = |x|e^{3x}$$

$$= xe^{3x},\ x>0$$

$$y(x) = \frac{1}{xe^{3x}}\int xe^{3x}\left(\frac{1}{x}e^{-3x}\right)dx$$

$$= \frac{1}{xe^{3x}}\int 1\,dx = \frac{1}{xe^{3x}}(x+C)$$

$$y(x) = e^{-3x} + \frac{C}{x}e^{-3x}$$

$$y(1) = 0 \Rightarrow$$

$$e^{-3} + Ce^{-3x} = 0$$

$$C = -1$$

$$\boxed{y(x) = e^{-3x}\left(1-\frac{1}{x}\right)}$$

6. (8 points) Analyze each initial value problem and determine whether we should expect a unique solution to exist through the given point.

(a) $(1-y)\dfrac{dy}{dx} = xy + 1, \quad y(2) = 1$

$$\frac{dy}{dx} = \frac{xy+1}{1-y}$$

$$f(x,y) = \frac{xy+1}{1-y}$$

$$f_y(x,y) = \frac{(1-y)(x) - (xy+1)(-1)}{(1-y)^2}$$

THESE ARE NOT CONTINUOUS AROUND $(2,1)$.

WE CANNOT EXPECT A UNIQUE SOL'N THROUGH $(2,1)$.

(b) $\dfrac{dy}{dx} - e^x y = x\cos^2 x, \quad y(0) = 0$

THIS IS A LINEAR FUNCTION WHOSE COEFFICIENT FUNCTIONS, $-e^x$ AND $x\cos^2 x$, ARE CONTINUOUS EVERYWHERE. WE EXPECT A UNIQUE SOLUTION DEFINED FOR ALL x.

7. (12 points) Solve: $\quad xy\,dx + (2x^2 + 3y^2)\,dy = 0, \quad y(1) = 1$

$M = xy$

$N = 2x^2 + 3y^2$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 4x$$

NOT EXACT! BUT... $\dfrac{x - 4x}{-xy} = \dfrac{3}{y}$ IS A FUNC OF ONLY y.

$$\mu(y) = e^{\int \frac{3}{y}\,dy} = e^{\ln|y|^3} = y^3, \; y > 0$$

$$xy^4\,dx + (2x^2y^3 + 3y^5)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 4xy^3 = \frac{\partial N}{\partial x} = 4xy^3 \quad \text{EXACT!}$$

$$\frac{\partial F}{\partial x} = xy^4 \Rightarrow F(x,y) = \tfrac{1}{2}x^2y^4 + g(y)$$

$$\frac{\partial F}{\partial y} = 2x^2y^3 + 3y^5 \Rightarrow F(x,y) = \tfrac{1}{2}x^2y^4 + \tfrac{1}{2}y^6 + h(x)$$

$$F(x,y) = \tfrac{1}{2}x^2y^4 + \tfrac{1}{2}y^6$$

$$F(1,1) = 1$$

SOL'N IS

$$\boxed{\tfrac{1}{2}x^2y^4 + \tfrac{1}{2}y^6 = 1}$$

8. (8 points) The following initial value problem arose while solving a mixing problem.

$$\frac{dA}{dt} = 8 - \frac{2A}{100+5t}, \quad A(0) = 1$$

Use Euler's method with $h = 1$ to approximate $A(3)$.

$A_0 = 1$

$t_0 = 0$

$A_1 = 1 + (1)\left(8 - \frac{2(1)}{100+5(0)}\right) = 8.98$

$t_1 = 1$

$A_2 = 8.98 + (1)\left(8 - \frac{2(8.98)}{100+5(1)}\right) = 16.80895238\ldots$

$t_2 = 2$

$A_3 = 16.8089\ldots + (1)\left(8 - \frac{2(16.8089\ldots)}{100+5(2)}\right) = 24.50333506\ldots$

$t_3 = 3$

$\boxed{A(3) \approx 24.5}$

9. (12 points) A large tank initially contains 75 gal of pure water. A salt solution containing 0.25 lb of salt per gallon enters the tank at 5 gal/min and is uniformly mixed. The mixed solution leaves the tank at 1 gal/min. Let $A(t)$ denote the amount of salt in the tank after t minutes. Set up and solve the appropriate initial value problem to determine $A(t)$. How much salt is in the tank after 20 minutes?

$$\frac{0.25\text{ lb}}{\text{GAL}} \cdot \frac{5\text{ GAL}}{\text{MIN}} = 1.25\ \frac{\text{lb}}{\text{MIN}}$$

$V(0) = 75 \qquad A(0) = 0$

$V(t) = 75 + 4t \qquad A(t) = ?$

$$\frac{1\text{ GAL}}{\text{MIN}} \quad \frac{A(t)}{V(t)}\frac{\text{lb}}{\text{GAL}}$$

$$\frac{dA}{dt} = 1.25 - \frac{A}{75+4t}$$

$$\frac{dA}{dt} + \frac{1}{75+4t}A = 1.25$$

$$\mu(t) = e^{\int \frac{1}{75+4t}dt} = e^{\frac{1}{4}\ln|75+4t|}$$

$$= (75+4t)^{1/4}$$

$$A(t) = (75+4t)^{-1/4}\int 1.25(75+4t)^{1/4}\,dt$$

$$= (75+4t)^{-1/4}\left[\frac{1.25}{5/4}\cdot\frac{1}{4}(75+4t)^{5/4} + C\right]$$

$$A(t) = 0.25(75+4t) + C(75+4t)^{-1/4}$$

$A(0) = 0 \Rightarrow$

$$0 = 18.75 + \frac{C}{\sqrt[4]{75}}$$

$$C = -18.75\sqrt[4]{75}$$
$$\approx -55.178$$

$A(20) \approx 23.112$ lb