Math 216 - Test 2a

October 20, 2015

Name ______________________

Score ____________

Show all work. Supply explanations where necessary.

1. (10 points) Solve: $y'' - 2y' - 15y = 0;\ y(0) = 3,\ y'(0) = -8$

CHAR EQN:

$r^2 - 2r - 15 = 0$

$(r-5)(r+3) = 0$

$r = 5,\ r = -3$

GEN SOL'N:

$y(x) = c_1 e^{5x} + c_2 e^{-3x}$

IC's:

$y(0) = c_1 + c_2 = 3$

$y'(x) = 5c_1 e^{5x} - 3c_2 e^{-3x}$

$y'(0) = 5c_1 - 3c_2 = -8$

$3(c_1 + c_2 = 3)$

$5c_1 - 3c_2 = -8$

$8c_1 = 1 \Rightarrow c_1 = \frac{1}{8}$

$c_2 = \frac{23}{8}$

$$\boxed{y(x) = \frac{1}{8}e^{5x} + \frac{23}{8}e^{-3x}}$$

2. (8 points) Find the general solution of $y^{(4)} - 3y'' = 0$.

CHAR EQN:

$r^4 - 3r^2 = 0$

$r^2(r^2 - 3) = 0$

$r = 0,\ r = 0,\ r = \sqrt{3},\ r = -\sqrt{3}$

$\{1, x, e^{\sqrt{3}x}, e^{-\sqrt{3}x}\}$

GEN SOL'N:

$$\boxed{y(x) = c_1 + c_2 x + c_3 e^{\sqrt{3}x} + c_4 e^{-\sqrt{3}x}}$$

3. (10 points) Solve the initial value problem.

$$x\frac{dy}{dx} = y + xe^{y/x}, \ y(1) = 1$$

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$u = \frac{y}{x}$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = u + e^{u}$$

$$x\frac{du}{dx} = e^{u}$$

$$e^{-u}\,du = \frac{1}{x}\,dx$$

$$-e^{-u} = \ln|x| + C$$

$$\ln|x| + e^{-y/x} = C$$

$$y(1) = 1 \Rightarrow e^{-1} = C$$

$$\ln|x| + e^{-y/x} = e^{-1}$$

OR

$$y(x) = -x\ln\left(e^{-1} - \ln|x|\right)$$

4. (8 points) Determine the recursive formulas for the Taylor method of order 2 for the IVP

$$\frac{dy}{dx} = x + y^2, \quad y(0) = 0.$$

$$f(x,y) = x + y^2$$

$$f'(x,y) = 1 + 2y\frac{dy}{dx} = 1 + 2y(x+y^2)$$

$$= 1 + 2xy + 2y^3$$

$$y_{N+1} = y_N + h(x_N + y_N^2) + \frac{h^2}{2}(1 + 2x_N y_N + 2y_N^3)$$

$$x_{N+1} = x_N + h$$

$$y_0 = 0$$

$$x_0 = 0$$

5. (16 points) An object is launched from the ground into the air so that its velocity, in meters per second, at any time t (in seconds) satisfies the initial value problem

$$v' = -0.5v - 9.8, \quad v(0) = 60.$$

Determine the function that gives the height of the object at time t. Then estimate when the object will hit the ground.

$$v' + 0.5v = -9.8$$

$$\mu(t) = e^{\int 0.5\,dt} = e^{0.5t}$$

$$v(t) = e^{-0.5t} \int -9.8 e^{0.5t}$$

$$= e^{-0.5t}\left(-19.6 e^{0.5t} + C\right)$$

$$v(0) = 60 \Rightarrow C = 79.6$$

$$v(t) = -19.6 + 79.6 e^{-0.5t}$$

$$x(t) = \int v(t)\,dt$$

$$x(t) = -19.6t - 159.2 e^{-0.5t} + C$$

$$x(0) = 0 \Rightarrow C = 159.2$$

$$\boxed{x(t) = -19.6t - 159.2 e^{-0.5t} + 159.2}$$

CALCULATOR'S solve gives $x(t) = 0$ WHEN $t = 7.9715\ldots$

$x(t) = 0$ AT ABOUT THE TIME WHEN

$$-19.6t + 159.2 = 0$$

OR $\boxed{t \approx 8 \text{ SEC}}$

6. (12 points) Consider the one-parameter family of curves described by

$$y = \frac{Cx}{1+x}. \quad \Rightarrow \quad C = \frac{y(1+x)}{x}$$

Find the family of orthogonal trajectories.

$$\frac{dy}{dx} = \frac{C(1+x) - Cx}{(1+x)^2} = \frac{C}{(1+x)^2} = \frac{y}{x(1+x)}$$

Neg. Recip ...

$$\frac{dy}{dx} = -\frac{x+x^2}{y}$$

$$-y\,dy = (x+x^2)\,dx$$

$$-\frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$$

$$\boxed{3x^2 + 3y^2 + 2x^3 = C}$$

7. (6 points) Consider the equation $xy'' - y' = 0$, $0 < x < \infty$.

(a) Verify that $y_1(x) = 1$ and $y_2(x) = x^2$ are solutions.

$y_1(x) = 1$
$y_1'(x) = 0$
$y_1''(x) = 0$

$xy_1'' - y_1' = 0 - 0 = 0$ ✓

$y_2(x) = x^2$
$y_2'(x) = 2x$
$y_2''(x) = 2$

$xy_2'' - y_2' = 2x - 2x = 0$ ✓

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(0, \infty)$.

$$\begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x \neq 0 \text{ on } (0,\infty)$$

(c) Using your results from above, state the general solution of the ODE.

$$\boxed{y(x) = c_1 + c_2 x^2}$$

Math 216 - Test 2b
October 20, 2015

Name key

Score

Show all work. Supply explanations where necessary.

1. (10 points) Solve: $\dfrac{dy}{dx} = y(xy^3 - 1),\ y(0) = 1$

$$\frac{dy}{dx} + y = xy^4$$

$$y^{-4}\frac{dy}{dx} + y^{-3} = x$$

$$u = y^{-3}$$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{du}{dx} = y^{-4}\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{du}{dx} + u = x$$

$$\frac{du}{dx} - 3u = -3x$$

$$\mu(x) = e^{\int -3\,dx} = e^{-3x}$$

$$u(x) = e^{3x}\int -3x e^{-3x}\,dx$$

$+$	$-3x$	e^{-3x}
$-$	-3	$-\frac{1}{3}e^{-3x}$
$+$	0	$\frac{1}{9}e^{-3x}$

$$= e^{3x}\left(xe^{-3x} + \frac{1}{3}e^{-3x} + C\right)$$

$$= x + \frac{1}{3} + Ce^{3x}$$

$$y^{-3} = x + \frac{1}{3} + Ce^{3x}$$

$y(0) = 1 \Rightarrow$

$$1 = \frac{1}{3} + Ce^{0}$$

$$\Rightarrow C = \frac{2}{3}$$

$$\boxed{y(x) = \frac{1}{\sqrt[3]{x + \frac{1}{3} + \frac{2}{3}e^{3x}}}}$$

2. (10 points) Referring to the IVP above, use a Runge-Kutta method of order 4 or 5 to estimate the x-value at which $y(x) = 0.5$. Start by using stepsize $h = 0.1$, then decrease your stepsize until you believe your solution has at least 3 correct digits. Show enough "work" to be worth 10 points!

USING RK5...

$h = 0.1 \Rightarrow$ $y(x) = 0.5$ FOR SOME X BETWEEN 0.7 & 0.8

$y(0.7) \approx 0.536$ $y(0.8) \approx 0.490$

$h = 0.01 \Rightarrow$ $y(x) = 0.5$ FOR SOME X BETWEEN 0.77 & 0.78

$y(0.77) \approx 0.504$ $y(0.78) \approx 0.499$

$h = 0.001 \Rightarrow$ $y(x) = 0.5$ FOR SOME X BETWEEN 0.778 & 0.779

$y(0.778) \approx 0.500$ $y(0.779) \approx 0.500$

$h = 0.0001 \Rightarrow$ $y(x) = 0.5$ FOR SOME X BETWEEN 0.7784 AND 0.7785

I'LL GO WITH $y(x) = 0.5$

$\Downarrow$

$x \approx 0.77845$

3. (10 points) Criminals in a boat are at the point $(1, 0)$ when the police shine a spotlight on them. The criminals evade the police by constantly moving counter-clockwise at a $45°$ angle from the light beam (which is following them). It turns out that the path of the criminals' boat satisfies the IVP

$$\frac{dy}{dx} = \frac{y/x + 1}{1 - y/x}, \quad y(1) = 0.$$

Solve the IVP. Then use the Implicit Equations Grapher (do a Google search) to graph the path.

$$u = \frac{y}{x}, \quad \frac{dy}{dx} = u + x\frac{du}{dx}, \quad u(1) = 0$$

$$u + x\frac{du}{dx} = \frac{u+1}{1-u}$$

$$x\frac{du}{dx} = \frac{u+1}{1-u} - u = \frac{1+u^2}{1-u}$$

$$\frac{1-u}{1+u^2}\,du = \frac{1}{x}\,dx$$

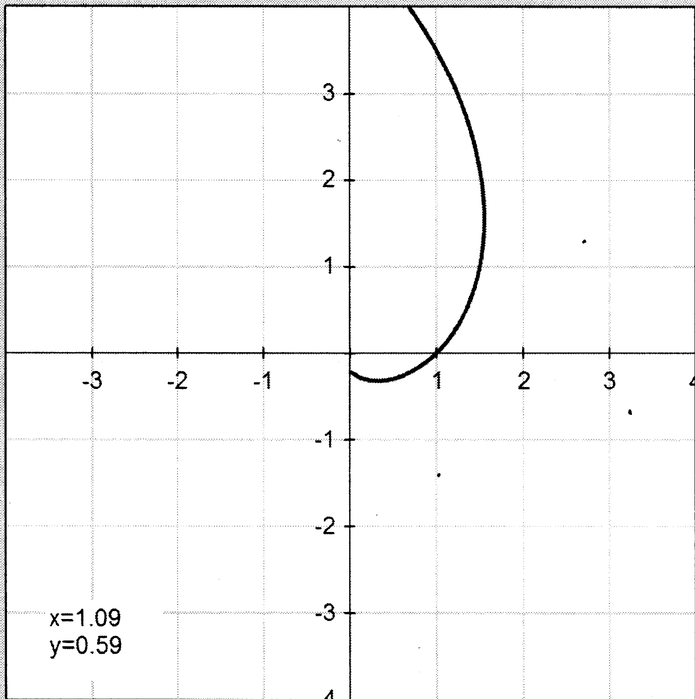
$$\int \frac{1}{1+u^2}\,du - \int \frac{u}{1+u^2}\,du = \int \frac{1}{x}\,dx$$

$$\tan^{-1} u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$$

$$u(1) = 0 \Rightarrow C = 0$$

$$\tan^{-1}(y/x) - \frac{1}{2}\ln(1 + y^2/x^2) = \ln x, \quad x > 0$$

SEE NEXT PAGE FOR GRAPH.



-4 xmin

xmax 4

4 ymax

ymin -4

Implicit Plotter

In the box below enter an equation in terms of x and y of the form:

f(x,y)=g(x,y)

For example:

x^2/9+y^2/4=1

Use ordinary calculator syntax when entering sides of the equation. Always use * for multiplication and parentheses around functions' arguments.

Mouse over the SYNTAX button below for a complete list of functions and syntax rules.

Enter your equation and x and y ranges in the range boxes. Then click the GRAPH button.

Enter an equation of the form f(x,y)=g(x,y):

atan(y/x)-ln(1+y^2/x^2)/2=ln(x)

GRAPH

Click the RESET button to reset the ranges and erase the graph. Mouse over the SYNTAX button for syntax rules.

RESET

SYNTAX

Math Applet by Doug Ensley and Barbara Kaskosz

www.flashandmath.com