Math 216 - Test 3a

November 18, 2015

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

   (a) $x'' + 8x = 0$

   NO DAMPING. SIMPLE HARMONIC MOTION

   (b) $x(t) = 6e^{-t/3}\sin(4t + \pi)$

   DAMPED OSCILLATIONS. UNDERDAMPED

   (c) $5x'' + 8x' + 2x = 0$

   $b^2 = 64$

   $4mk = 40$

   $b^2 > 4mk$ OVERDAMPED

   (d) $x(t) = \underbrace{7e^{-3t} + 9te^{-3t}}$

   CHARACTERISTIC POLY HAS REPEATED ZERO

   CRITICALLY DAMPED

2. (6 points) Consider the following equation:

$$y'' + 4y = x\cos x + \cos 2x.$$

Solve the corresponding homogeneous equation. Then find the appropriate form of the particular solution for the nonhomogeneous equation. Do not solve for the undetermined coefficients.

HOMO EQUATION:

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm 2i$

$\boxed{y_h(x) = c_1\cos 2x + c_2\sin 2x}$

NONHOMO EQUATION:

$g(x) = x\cos x + \cos 2x$

$\Downarrow$

$\boxed{y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x + Ex\cos 2x + Fx\sin 2x}$

3. (6 points) Solve the following initial value problem.

$$2y'' - 2y' + y = 0; \quad y(0) = -1,\ y'(0) = 0$$

$2r^2 - 2r + 1 = 0$

$r = \dfrac{2 \pm \sqrt{4 - 4(2)(1)}}{4} = \dfrac{2 \pm \sqrt{-4}}{4}$

$r = \frac{1}{2} \pm \frac{1}{2} i$

$y(t) = c_1 e^{t/2} \cos \frac{t}{2} + c_2 e^{t/2} \sin \frac{t}{2}$

$y(0) = -1 \Rightarrow c_1 = -1$

$y'(t) = \frac{c_1}{2} e^{t/2} \cos \frac{t}{2} - \frac{c_1}{2} e^{t/2} \sin \frac{t}{2}$

$\quad + \frac{c_2}{2} e^{t/2} \sin \frac{t}{2} + \frac{c_2}{2} e^{t/2} \cos \frac{t}{2}$

$y'(0) = 0 \Rightarrow \frac{c_1}{2} + \frac{c_2}{2} = 0 \Rightarrow c_1 = -c_2$

$\Rightarrow c_2 = 1$

$$\boxed{y(t) = -e^{t/2} \cos \frac{t}{2} + e^{t/2} \sin \frac{t}{2}}$$

4. (12 points) Solve the following initial value problem.

$$y'' - 2y' - 3y = 6xe^{2x}; \quad y(0) = 1,\ y'(0) = -1$$

Homo eqn:

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3,\ r = -1$

$y_h(x) = c_1 e^{3x} + c_2 e^{-x}$

NonHomo eqn:

$g(x) = 6xe^{2x}$

$y_p(x) = (Ax+B)e^{2x}$

$y_p'(x) = Ae^{2x} + 2Axe^{2x} + 2Be^{2x}$

$y_p''(x) = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} + 4Be^{2x}$

$y_p'' - 2y_p' - 3y_p = 6xe^{2x}$

$\Downarrow$

$(4A + 4Ax + 4B) - 2(A + 2Ax + 2B) - 3(Ax + B) = 6x$

$$\left.\begin{array}{l} 2A - 3B = 0 \\ -3Ax = 6x \end{array}\right\} \quad \begin{array}{l} A = -2 \\ B = -\frac{4}{3} \end{array}$$

$y_p(x) = \left(-2x - \frac{4}{3}\right)e^{2x}$

$y(x) = c_1 e^{3x} + c_2 e^{-x} - \left(2x + \frac{4}{3}\right)e^{2x}$.

$y(0) = 1 \Rightarrow c_1 + c_2 - \frac{4}{3} = 1$

$c_1 + c_2 = \frac{7}{3}$

$y'(x) = 3c_1 e^{3x} - c_2 e^{-x} - 2\left(2x + \frac{4}{3}\right)e^{2x} - 2e^{2x}$

$y'(0) = -1 \Rightarrow 3c_1 - c_2 - \frac{8}{3} - 2 = -1$

$3c_1 - c_2 = \frac{11}{3}$

$c_1 + c_2 = \frac{7}{3}$

$3c_1 - c_2 = \frac{11}{3}$

$4c_1 = 6 \qquad c_1 = \frac{3}{2}$

$c_2 = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$

$$\boxed{y(x) = \frac{3}{2}e^{3x} + \frac{5}{6}e^{-x} - \left(2x + \frac{4}{3}\right)e^{2x}}$$

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Show all work to receive full credit. Supply explanations where necessary. YOU MUST WORK INDIVIDUALLY ON THIS EXAM. NO CREDIT WILL BE GIVEN FOR GROUP WORK.

1. (12 points) Solve: $y'' + 2y' + y = e^{-x} + e^{-x}\ln x$

Homo eq'n:

$y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1,\ r = -1$

$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$

Nonhomo eq'n:

$g(x) = e^{-x} + e^{-x}\ln x$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$

$$V_1(x) = \int -\frac{(e^{-x} + e^{-x}\ln x)\,xe^{-x}}{e^{-2x}}\,dx$$

$$= \int -(x + x\ln x)\,dx$$

$$= -\left(\frac{x^2}{2} + \frac{1}{2}x^2\ln x - \frac{x^2}{4}\right)$$

$$= -\frac{x^2}{4} - \frac{x^2}{2}\ln x$$

$$V_2(x) = \int \frac{(e^{-x} + e^{-x}\ln x)\,e^{-x}}{e^{-x}}\,dx$$

$$= \int (1 + \ln x)\,dx$$

$$= x + x\ln x - x = x\ln x$$

$$y_p(x) = \left(-\frac{x^2}{4} - \frac{x^2}{2}\ln x\right)e^{-x} + (x\ln x)\,xe^{-x}$$

$$= -\frac{1}{4}x^2e^{-x} + \frac{1}{2}x^2e^{-x}\ln x$$

$$y(x) = \left(c_1 + c_2 x - \frac{1}{4}x^2 + \frac{1}{2}x^2\ln x\right)\cdot(e^{-x})$$

2. (14 points) Solve: $y'' - 2y' + 5y = 5x + 8 + x\sin x$

Homo eq'n:

$y'' - 2y' + 5y = 0$

$r^2 - 2r + 5 = 0$

$r^2 - 2r + 1 = -4$

$(r-1)^2 = -4$

$r - 1 = \pm 2i$

$r = 1 \pm 2i$

$y_h(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x$

NonHomo #1:

$g(x) = 5x + 8$

$y_p(x) = Ax + B$

$y_p'(x) = A$

$y_p''(x) = 0$

$-2A + 5(Ax + B) = 5x + 8$

$A = 1$

$B = 2$

$y_p(x) = x + 2$

NonHomo #2:

$g(x) = x \sin x$

$y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x$

$y_p'(x) = A\cos x - (Ax + B)\sin x + C\sin x + (Cx + D)\cos x$

$y_p''(x) = -A\sin x - A\sin x - (Ax + B)\cos x + C\cos x - (Cx + D)\sin x + C\cos x$

$y_p'' - 2y_p' + 5y_p = x\sin x$

$\Downarrow$

$\cos x:\ 2C - B - 2(A + D) + 5(B) = 0$

$x\cos x:\ -A - 2(C) + 5(A) = 0$

$\sin x:\ -2A - D - 2(-B + C) + 5(D) = 0$

$x\sin x:\ -C - 2(-A) + 5(C) = 1$

$-2A + 4B + 2C - 2D = 0$

$4A - 2C = 0$

$-2A + 2B - 2C + 4D = 0$

$2A + 4C = 1$

$\Rightarrow$ $A = 0.1$, $B = 0.02$, $C = 0.2$, $D = 0.14$

$y(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x + x + 2 + (0.1x + 0.02)\cos x + (0.2x + 0.14)\sin x$

3. (12 points) Solve the following nonhomogeneous, Cauchy-Euler equation.

$$x^2 \frac{d^2y}{dx^2} + 9x\frac{dy}{dx} - 20y = \frac{5}{x^3}$$

$x = e^t \Rightarrow y'' + 8y' - 20y = 5e^{-3t}$

Homo eqn: $y'' + 8y' - 20y = 0$

$r^2 + 8r - 20 = 0$

$(r+10)(r-2) = 0$

$y_h(t) = c_1 e^{-10t} + c_2 e^{2t}$

Nonhomo: $g(t) = 5e^{-3t}$

$y_p(t) = Ae^{-3t}$

$y_p' = -3Ae^{-3t}$

$y_p'' = 9Ae^{-3t}$

$y'' + 8y' - 20y = 5e^{-3t}$

$\Downarrow$

$9A + 8(-3A) - 20A = 5$

$-35A = 5$

$A = -\frac{1}{7}$

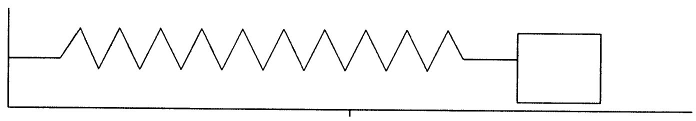
$y_p(t) = -\frac{1}{7}e^{-3t}$

$y(t) = c_1 e^{-10t} + c_2 e^{2t} - \frac{1}{7}e^{-3t}$

$\Downarrow$

$y(x) = c_1 x^{-10} + c_2 x^2 - \frac{1}{7}x^{-3}$

4. (12 points) A 1-kg mass is attached to a spring with spring constant 16 N/m. The damping constant for the system is 10 N-sec/m. The mass is moved 1 m to the right of equilibrium (stretching the spring) and pushed to the left at 12 m/sec. Find the equation of motion. Is the system underdamped, overdamped, or critically damped? How do you know?



$$x'' + 10x' + 16x = 0, \quad x(0) = 1, \quad x'(0) = -12$$

$$r^2 + 10r + 16 = 0$$

$$(r+8)(r+2) = 0$$

$$x(t) = c_1 e^{-8t} + c_2 e^{-2t}$$

$$x(0) = 1 \Rightarrow c_1 + c_2 = 1$$

$$x'(t) = -8c_1 e^{-8t} - 2c_2 e^{-2t}$$

$$x'(0) = -12 \Rightarrow -8c_1 - 2c_2 = -12$$

$$c_1 + c_2 = 1$$

$$-8c_1 - 2c_2 = -12$$

$$6c_2 = -4$$

$$c_2 = -\frac{2}{3}$$

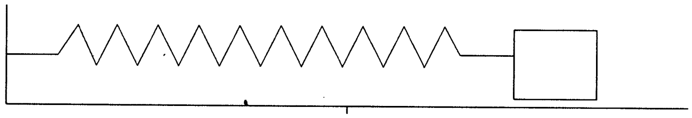
$$c_1 = \frac{5}{3}$$

$$\boxed{x(t) = \frac{5}{3}e^{-8t} - \frac{2}{3}e^{-2t}}$$

SYSTEM IS OVERDAMPED.

(NO OSCILLATIONS)

5. (18 points) A 4-kg mass is attached to a spring with spring constant 169 N/m. The damping constant for the system is 12.8 N-sec/m. The mass is moved 0.1 m to the right of equilibrium (stretching the spring) and pushed to the right at 1.3 m/sec. Find the equation of motion. Write your final result in terms of a single trig function with phase shift. Then determine when the mass passes through equilibrium for the first time.



$x = 0$
(Equilibrium)

$4x'' + 12.8x' + 169x = 0, \quad x(0) = 0.1, \quad x'(0) = 1.3$

$4r^2 + 12.8r + 169 = 0$

$r = \dfrac{-12.8 \pm \sqrt{12.8^2 - 4(4)(169)}}{8}$

$= \dfrac{-12.8}{8} \pm \dfrac{50.4}{8} i = -1.6 \pm 6.3i$

$x(t) = A e^{-1.6t} \sin(6.3t + \varphi)$

$x(0) = 0.1 \Rightarrow A \sin\varphi = 0.1$

$x'(t) = -1.6 A e^{-1.6t} \sin(6.3t + \varphi)$
$\quad + 6.3 A e^{-1.6t} \cos(6.3t + \varphi)$

$x'(0) = -1.6 A \sin\varphi + 6.3 A \cos\varphi = 1.3$
$6.3 A \cos\varphi = 1.46$

$A \sin\varphi = \frac{1}{10} \qquad A\cos\varphi = \frac{1.46}{6.3}$

$A \approx 0.2524$

$\tan\varphi \approx 0.4315$
$\Rightarrow \varphi \approx 0.40737$

$x(t) = 0.2524\, e^{-1.6t} \sin(6.3t + 0.40737)$

$6.3t + 0.40737 = k\pi$

$\Rightarrow t = \dfrac{k\pi - 0.40737}{6.3}$

$k = 1 \Rightarrow \boxed{t = 0.434 \text{ sec}}$

6. (2 points) Determine the resonance (angular) frequency for the system in the problem above.

$\gamma_r = \sqrt{\dfrac{169}{4} - \dfrac{(12.8)^2}{2(4)^2}} \approx 6.093$