Math 216 - Final Exam

December 9, 2015

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Solve the following initial value problem:

$$\frac{dy}{dx} = 3xy, \quad y(0) = 5$$

$$\frac{1}{y}\,dy = 3x\,dx$$

$$\ln|y| = \frac{3}{2}x^2 + C_1$$

$$|y| = C_2 e^{3x^2/2}$$

$$y = C_3 e^{3x^2/2}$$

$$y(0) = 5 \Rightarrow C_3 = 5$$

$$\boxed{y(x) = 5e^{3x^2/2}}$$

2. (10 points) Find the orthogonal trajectories for the one-parameter family of curves $y = 3x^2 + C$.

$$\frac{dy}{dx} = 6x$$

ORTHO TRAJS SATISFY

$$\frac{dy}{dx} = \frac{-1}{6x}$$

$$-6\,dy = \frac{1}{x}\,dx$$

$$-6y = \ln|x| + C$$

$$\boxed{\ln|x| + 6y = C}$$

3. (6 points) Consider the initial value problem $y' = xy$, $y(1) = 2$. Use Euler's method with $h = 0.5$ to approximate $y(2)$.

$y_0 = 2,\ x_0 = 1$

$y_1 = 2 + 0.5(1)(2) = 3$

$x_1 = 1.5$

$y_2 = 3 + 0.5(1.5)(3) = 5.25$

$x_2 = 2$

$y(2) \approx 5.25$

4. (8 points) Determine the recursive formulas for the Taylor method of order 3 for the IVP

$$\frac{dy}{dx} = x + 2y, \quad y(0) = 1.$$

$y_{N+1} = y_N + h f(x_N, y_N) + \frac{h^2}{2} f'(x_N, y_N) + \frac{h^3}{6} f''(x_N, y_N)$

$$\left.\begin{aligned} f(x,y) &= x + 2y \\ f'(x,y) &= 1 + 2y' = 1 + 2x + 4y \\ f''(x,y) &= 2 + 4y' = 2 + 4x + 8y \end{aligned}\right\}$$

$$y_{N+1} = y_N + h(x_N + 2y_N) + \frac{h^2}{2}(1 + 2x_N + 4y_N) + \frac{h^3}{6}(2 + 4x_N + 8y_N)$$

$$x_{N+1} = x_N + h$$

5. (12 points) Solve the initial value problem

$$y'' + 2y' + 4y = 0; \quad y(0) = 2,\ y'(0) = -1$$

$r^2 + 2r + 4 = 0$

$r^2 + 2r + 1 = -3$

$(r+1)^2 = -3$

$r = -1 \pm \sqrt{3}\, i$

$y(x) = c_1 e^{-x} \cos\sqrt{3}x + c_2 e^{-x} \sin\sqrt{3}x$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(x) = -c_1 e^{-x}\cos\sqrt{3}x - \sqrt{3}c_1 e^{-x}\sin\sqrt{3}x$
$\quad - c_2 e^{-x}\sin\sqrt{3}x + \sqrt{3}c_2 e^{-x}\cos\sqrt{3}x$

$y'(0) = -1 \Rightarrow -c_1 + \sqrt{3}c_2 = -1 \Rightarrow \sqrt{3}c_2 = 1 \Rightarrow c_2 = \frac{1}{\sqrt{3}}$

$$y(x) = 2e^{-x}\cos\sqrt{3}x + \frac{1}{\sqrt{3}}e^{-x}\sin\sqrt{3}x$$

6. (12 points) Find the general solution of the ODE

$$(x^2+1)y' + xy = 2x.$$

$$y' + \frac{x}{x^2+1} y = \frac{2x}{x^2+1}$$

$$\mu(x) = e^{\int \frac{x}{x^2+1} dx} = e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1}$$

$$y = \frac{1}{\sqrt{x^2+1}} \int \frac{2x}{x^2+1} \sqrt{x^2+1}\, dx = \frac{1}{\sqrt{x^2+1}} \int \frac{2x}{\sqrt{x^2+1}}\, dx$$

$$\int \frac{2x}{\sqrt{x^2+1}}\, dx = \int u^{-1/2}\, du = 2u^{1/2}$$

$$u = x^2+1$$

$$du = 2x\,dx$$

$$y(x) = \frac{1}{\sqrt{x^2+1}}\left(2\sqrt{x^2+1} + C\right)$$

$$y(x) = 2 + \frac{C}{\sqrt{x^2+1}}$$

7. (12 points) Find an integrating factor for the differential equation

$$(x + xy^3)\,dx + 3y^2\,dy = 0.$$

Then use your integrating factor to solve the equation.

$$\frac{\partial M}{\partial y} = 3xy^2$$

$$\frac{\partial N}{\partial x} = 0$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3xy^2}{3y^2} = x \Rightarrow \mu(x) = e^{\int x\,dx} = e^{x^2/2}$$

$$xe^{x^2/2} + xe^{x^2/2} y\, dx + 3y^2 e^{x^2/2}\, dy = 0$$

$$\frac{\partial F}{\partial x} = xe^{x^2/2} + xe^{x^2/2}y^3 \Rightarrow F(x,y) = e^{x^2/2} + e^{x^2/2}y^3 + g(y)$$

$$\frac{\partial F}{\partial y} = 3y^2 e^{x^2/2} \Rightarrow F(x,y) = y^3 e^{x^2/2} + h(x)$$

$$e^{x^2/2} + y^3 e^{x^2/2} = C$$

8. (16 points) Solve the following initial value problem:

$$y'' - 5y' + 4y = 2e^{4x}; \quad y(0) = 1,\ y'(0) = 2$$

Homo eqn: $y'' - 5y' + 4y = 0$

$r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$r = 4,\ r = 1$

$y_h(x) = c_1 e^x + c_2 e^{4x}$

NonHomo eqn: $g(x) = 2e^{4x}$

$y_p(x) = Axe^{4x}$

$y_p' = Ae^{4x} + 4Axe^{4x}$

$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$

$y_p'' - 5y_p' + 4y_p = 2e^{4x}$

$\Downarrow$

$(\cancel{16Axe^{4x}} + 8Ae^{4x}) - 5(Ae^{4x} + \cancel{4Axe^{4x}})$

$+ \cancel{4Axe^{4x}} = 2e^{4x}$

$\Rightarrow \quad 3Ae^{4x} = 2e^{4x}$

$\Rightarrow A = \frac{2}{3}$

$y_p(x) = \frac{2}{3}xe^{4x}$

$y(x) = c_1 e^x + c_2 e^{4x} + \frac{2}{3}xe^{4x}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 2 \Rightarrow c_1 + 4c_2 + \frac{2}{3} = 2$

$c_1 + c_2 = 1$

$c_1 + 4c_2 = \frac{4}{3}$

$3c_2 = \frac{1}{3}$

$c_2 = \frac{1}{9}$

$c_1 = \frac{8}{9}$

$y(x) = \frac{8}{9}e^x + \frac{1}{9}e^{4x} + \frac{2}{3}xe^{4x}$

9. (16 points) Consider the initial value problem

$$x'' + 9x = e^t; \quad x(0) = 2, \ x'(0) = 1.$$

(a) Compute the Laplace transform of both sides of the equation. Then solve for $X(s)$, the Laplace transform of $x(t)$.

$$s^2 X - 2s - 1 + 9X = \frac{1}{s-1}$$

$$(s^2+9)X = \frac{1}{s-1} + 2s + 1$$

$$X(s) = \frac{\frac{1}{s-1} + 2s + 1}{s^2+9}$$

(b) After expanding your solution above, you would find that

$$X(s) = \frac{1}{10}\left(\frac{1}{s-1}\right) + \frac{19}{10}\left(\frac{s}{s^2+9}\right) + \frac{9}{10}\left(\frac{1}{s^2+9}\right).$$

Compute the inverse transform of $X(s)$ to determine $x(t)$.

$$X(s) = \frac{1}{10}\frac{1}{s-1} + \frac{19}{10}\frac{s}{s^2+9} + \frac{9}{30}\frac{3}{s^2+9}$$

$$\Downarrow$$

$$x(t) = \frac{1}{10}e^t + \frac{19}{10}\cos 3t + \frac{9}{30}\sin 3t$$

10. (16 points) Find the general solution of $y'' + 4y = \sec 2x$.

Homo eqn: $y''+4y=0$

$r^2+4=0$

$r=\pm 2i$

$y_h(x) = C_1 \cos 2x + C_2 \sin 2x$

NonHomo: $g(x) = \sec 2x$

$y_1(x) = \cos 2x \quad y_2(x) = \sin 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$V_1(x) = \int \frac{-\sec 2x \sin 2x}{2}\,dx = -\frac{1}{2}\int \tan 2x\,dx$$

$$= \frac{1}{4}\ln|\cos 2x|$$

$$V_2(x) = \int \frac{\sec 2x \cos 2x}{2}\,dx = \frac{1}{2}\int dx = \frac{1}{2}x$$

$$y_p(x) = \frac{1}{4}\cos 2x \ln|\cos 2x| + \frac{1}{2}x\sin 2x$$

$$y(x) = C_1\cos 2x + C_2 \sin 2x + \frac{1}{4}\cos 2x \ln|\cos 2x| + \frac{1}{2}x \sin 2x$$

11. (12 points) An object is launched from the ground into the air so that its velocity, in feet per second, at any time t (in seconds) satisfies the initial value problem

$$v' = -27v - 32, \quad v(0) = 20.$$

Determine the function that gives the height of the object at time t.

$$v' + 27v = -32$$

$$\mu(t) = e^{\int 27\,dt} = e^{27t}$$

$$v = e^{-27t}\int -32e^{27t}\,dt$$

$$= e^{-27t}\left[-\frac{32}{27}e^{27t} + C\right]$$

$$v(t) = -\frac{32}{27} + Ce^{-27t}$$

$$v(0) = 20 \Rightarrow -\frac{32}{27} + C = 20 \Rightarrow C = \frac{572}{27}$$

$$v(t) = -\frac{32}{27} + \frac{572}{27}e^{-27t}$$

$$x(t) = C - \frac{32}{27}t - \frac{572}{(27)^2}e^{-27t}$$

$$x(0) = 0 \Rightarrow C = \frac{572}{27^2} = \frac{572}{729}$$

$$x(t) = \frac{572}{729} - \frac{32}{27}t - \frac{572}{729}e^{-27t}$$

– OR –

$$x(t) = 0.785 - 1.185t - 0.785e^{-27t}$$

12. (20 points) Consider the following system of ODEs.

$$x' + y' - x = 5$$

$$x' + y' + y = 1$$

(a) Write the system in operator notation and then determine the number of arbitrary constants in the general solution.

$$(D-1)x + Dy = 5$$
$$Dx + (D+1)y = 1$$

$$(D-1)(D+1) - D^2 = D^2 - 1 - D^2 = -1$$

$\Rightarrow$ NO ARBITRARY CONSTANTS

(b) Use any method to solve the system.

SUBTRACT EQUATIONS TO GET

$$-x - y = 4 \quad \text{OR} \quad y = -x - 4$$

SUBS INTO 1^{ST} EQN

$$x' + (-x-4)' - x = 5$$

$$x' - x' - x = 5$$

$$x = -5$$

$$y = -(-5) - 4 = 1$$

SOLUTION...

$$x(t) = -5$$
$$y(t) = 1$$