A *differential equation* is an equation containing one or more derivatives of an unknown function.

$$\frac{dP}{dt} = kP$$

$$m\,\frac{d^2x}{dt^2} = -mg$$

$$xy' + y = e^{-x^2}$$

$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{1}{C}\,Q = E$$

To solve a differential equation means to find the unknown function(s).

If an equation involves the derivative of one variable with respect to another, then the former is the *dependent variable* and the latter is the *independent variable*.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an *ordinary differential equation*.

A differential equation involving partial derivatives with respect to more than one independent variable is called an *partial differential equation*.

Classify each equation.

$$3\,\frac{d^2x}{dt^2} + 4\,\frac{dx}{dt} + 9x = 2\cos 3t$$

$$y'' - yy' = \tan x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right)^2 = e^x$$

$$\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} = x - 2y$$

$$y'' + y = x^3$$

$$\frac{d^2 s}{dt^2} + s^3 = 0$$

$$xy\,dx + (x^2 - y + 3)\,dy = 0$$

The *order* of a differential equation is the order of the highest-order derivatives that appear in the equation.

An ordinary differential equation is *linear* if it can be written in the form

$$a_n(x)\,\frac{d^n y}{dx^n} + \cdots + a_1(x)\,\frac{dy}{dx} + a_0(x)\,y = F(x),$$

where the coefficient functions and the right-hand side depend only on the independent variable x.

If an ordinary differential equation is not linear, it is called *nonlinear*.

Suppose you are given an ordinary differential equation with dependent variable y and independent variable x...

If the function $y = f(x)$ satisfies the differential equation for all x in an interval I, then $f(x)$ is called an *explicit solution* of the equation on I.

If a solution is defined on an interval I by an expression of the form $G(x, y) = 0$, then the solution is called an *implicit solution* on I.

Furthermore, if an explicit solution involves only elementary functions, then it is a *closed-form solution*.

$P(t) = Ce^{kt}$ is an explicit solution of

$$\frac{dP}{dt} = kP.$$

$xy = c + \ln y$ is an implicit solution of

$$y'(1 - xy) = y^2.$$

An explicit solution of $xy' + y = e^{-x^2}$ is

$$y(x) = \frac{1}{x} \int_0^x e^{-t^2}\, dt + \frac{C}{x},$$

but this solution is not closed-form.

An *initial value problem* for an nth order ordinary differential equation is a problem of finding a solution $y(x)$ that further satisfies n initial conditions of the form

$$y(x_0) = y_0,\, y'(x_0) = y_1,\, \ldots,\, y^{(n-1)}(x_0) = y_{n-1}$$

For example,

$$2\ddot{z} + 7\dot{z} - 4z = 0; \quad z(0) = 0,\, \dot{z}(0) = 9$$

And its solution is

$$z(t) = 2e^{t/2} - 2e^{-4t}$$

Existence/Uniqueness

Suppose we are given the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

If f and $\partial f / \partial y$ are continuous in a rectangle

$$\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$$

containing (x_0, y_0), then the IVP has a unique solution on the interval $(x_0 - h, x_0 + h)$ for some positive number h.

Unfortunately, this result only tells us that the IVP has a unique solution in a neighborhood of x_0. We have no idea how big that neighborhood actually is.

If the DE has a special form, we may be able to say more.

A *direction field* or *slope field* for a differential equation is a plot of short line segments drawn at various points in the xy-plane showing the slopes of the solution curves at those points.