Math 216 - Test 1
February 15, 2012

Name Key

Score

Show all work. Supply explanations when necessary. Give explicit solutions when possible.

1. (8 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $vv''' = x$ ORDINARY, NONLINEAR, 3RD ORDER

   (b) $y'' - 6y' + 12y = t^2 \sin t$

   ORDINARY, LINEAR, 2ND ORDER

   (c) $\dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} + u = 6y^2 - x$

   PARTIAL, LINEAR, 2ND ORDER

   (d) $(x^2 + y^2)\, dx + 2xy\, dy = 0$

   ORDINARY, NONLINEAR, 1ST ORDER

2. (12 points) Solve the following initial value problem:

$$(x^2 + 9)\frac{dy}{dx} + xy = 0, \qquad y(0) = 1.$$

$$\frac{dy}{dx} + \frac{x}{x^2+9}\, y = 0$$

$$\mu(x) = e^{\int \frac{x}{x^2+9}\,dx} = e^{\frac{1}{2}\ln|x^2+9|} = \sqrt{x^2+9}$$

$$\sqrt{x^2+9}\; y = \int 0\, dx$$

$$y = \frac{c}{\sqrt{x^2+9}}$$

$y(0) = 1 \Rightarrow C = 3$

$$\boxed{y = \frac{3}{\sqrt{x^2+9}}}$$

3. (12 points) Consider the following initial value problem:

$$\frac{dy}{dx} = \frac{x^3+1}{y+2}, \quad y(0) = 2$$

(a) Is the DE separable, linear, exact, or homogeneous? Your answer should correspond with your solution method.

$$(y+2)\,dy = (x^3+1)\,dx$$

SEPARABLE

(b) Solve the initial value problem.

$$\frac{1}{2}y^2 + 2y = \frac{1}{4}x^4 + x + C$$

$y(0) = 2 \Rightarrow$

$$\frac{1}{2}(2)^2 + 2(2) = \frac{1}{4}(0) + (0) + C$$

$$6 = C$$

$$\frac{1}{2}y^2 + 2y = \frac{1}{4}x^4 + x + 6$$

$$\boxed{2y^2 + 8y = x^4 + 4x + 24}$$

(c) Is your solution explicit or implicit?

My SOLUTION IS IMPLICIT.

4. (12 points) Consider the following differential equation:

$$\frac{dy}{dx} = \frac{x^2 - y^2}{3xy}$$

(a) Is the DE separable, linear, exact, or homogeneous? Your answer should correspond with your solution method.

$$3\frac{dy}{dx} = \frac{x}{y} - \frac{y}{x}$$ Homogeneous

(b) Solve the equation.

$$u = \frac{y}{x}$$

$$y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$3u + 3x\frac{du}{dx} = \frac{1}{u} - u$$

$$3x\frac{du}{dx} = \frac{1}{u} - 4u$$

$$3x\frac{du}{dx} = \frac{1-4u^2}{u}$$

$$\frac{u}{1-4u^2}\,du = \frac{1}{3x}\,dx$$

$$-\frac{1}{8}\ln|1-4u^2| = \frac{1}{3}\ln|x| + C_1$$

Exponentiate both sides...

$$|1-4u^2|^{-1/8} = C_2|x|^{1/3}$$

$$|1-4u^2| = \frac{C_3}{x^{8/3}}$$

$$\left|1-4\left(\frac{y}{x}\right)^2\right| = \frac{C_3}{x^{8/3}}$$

Mult both sides by x^2...

$$\boxed{x^2 - 4y^2 = \frac{C_4}{x^{2/3}}}$$

(c) Is your solution explicit or implicit?

My solution is implicit.

5. (12 points) Consider the following initial value problem:

$$\overbrace{[2x + y^2 - \cos(x+y)]}^{M(x,y)}\, dx + \overbrace{[2xy - \cos(x+y) - e^y]}^{N(x,y)}\, dy = 0, \quad y(\pi) = 0$$

(a) Is the DE separable, linear, exact, or homogeneous? Your answer should correspond with your solution method.

$$\frac{\partial M}{\partial y} = 2y + \sin(x+y) = \frac{\partial N}{\partial x} = 2y + \sin(x+y) \qquad \text{EXACT!}$$

(b) Solve the initial value problem.

$$\frac{\partial F}{\partial x} = 2x + y^2 - \cos(x+y) \Rightarrow F(x,y) = x^2 + xy^2 - \sin(x+y) + g(y)$$

$$\frac{\partial F}{\partial y} = 2xy - \cos(x+y) - e^y \Rightarrow F(x,y) = xy^2 - \sin(x+y) - e^y + h(x)$$

$$F(x,y) = x^2 + xy^2 - \sin(x+y) - e^y$$

$$F(\pi, 0) = \pi^2 - 1$$

SOLUTION IS

$$x^2 + xy^2 - \sin(x+y) - e^y = \pi^2 - 1$$

(c) Is your solution explicit or implicit?

MY SOLUTION IS IMPLICIT.

6. (9 points) Use Euler's method with $h = 0.1$ to approximate the value of $y(0.3)$ for the initial value problem $y' = (4x + y + 2)^2$, $y(0) = 0$.

$y_0 = 0$

$x_0 = 0$

$f(x,y) = (4x+y+2)^2$

$y_1 = y_0 + h f(x_0, y_0) = 0.1(2)^2 = 0.4$

$x_1 = 0.1$

$y_2 = y_1 + h f(x_1, y_1) = 0.4 + 0.1(0.4 + 0.4 + 2)^2 = 1.184$

$x_2 = 0.2$

$y_3 = y_2 + h f(x_2, y_2) = 1.184 + 0.1(0.8 + 1.184 + 2)^2 = 2.7712256$

$x_3 = 0.3$

$$\boxed{y(0.3) \approx 2.77}$$

7. (12 points) Use the substitution $u = 4x + y + 2$ to find the exact solution of the initial value problem above. Compare the exact value of $y(0.3)$ to your approximate value.

$u = 4x + y + 2$

$\frac{du}{dx} = 4 + \frac{dy}{dx}$

$\frac{du}{dx} - 4 = u^2$

$\frac{du}{dx} = u^2 + 4$

$\frac{1}{u^2+4}\,du = dx$

$\frac{1}{2}\tan^{-1}\frac{u}{2} = x + C$

$\tan^{-1}\frac{u}{2} = 2x + C$

$\frac{u}{2} = \tan(2x + C)$

$u = 2\tan(2x + C)$

$4x + y + 2 = 2\tan(2x + C)$

$y = 2\tan(2x + C) - 4x - 2$

$y(0) = 0 \Rightarrow 2\tan(C) - 2 = 0$

$C = \frac{\pi}{4}$

$$\boxed{y = 2\tan\left(2x + \frac{\pi}{4}\right) - 4x - 2}$$

$y(0.3) = 7.4637\ldots$

YIKES! APPROX ABOVE IS NOT EVEN CLOSE.

8. (13 points) Consider the following differential equation:

$$(y^3 + 4e^x y)\,dx + (2e^x + 3y^2)\,dy = 0$$

(a) Find the slope of the solution curve at the point $(0, 2)$.

At $(0,2)$: $(8 + 4e^0(2))\,dx + (2e^0 + 3(2)^2)\,dy = 0$

$16\,dx + 14\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{16}{14} = \boxed{-\frac{8}{7}}$

(b) Use the test for exactness to show that the equation is NOT exact.

$M(x,y) = y^3 + 4e^x y$ $\qquad \frac{\partial M}{\partial y} = 3y^2 + 4e^x$ $\qquad$ NOT

$N(x,y) = 2e^x + 3y^2$ $\qquad \neq \frac{\partial N}{\partial x} = 2e^x$ $\qquad$ EXACT!

(c) Multiply each side of the equation by e^x, and show that the new (equivalent) equation is exact.

$(y^3e^x + 4e^{2x}y)\,dx + (2e^{2x} + 3y^2e^x)\,dy = 0$

$\frac{\partial M}{\partial y} = 3y^2e^x + 4e^{2x} = \frac{\partial N}{\partial x} = 4e^{2x} + 3y^2e^x$ $\qquad$ EXACT!

(d) Solve the equation.

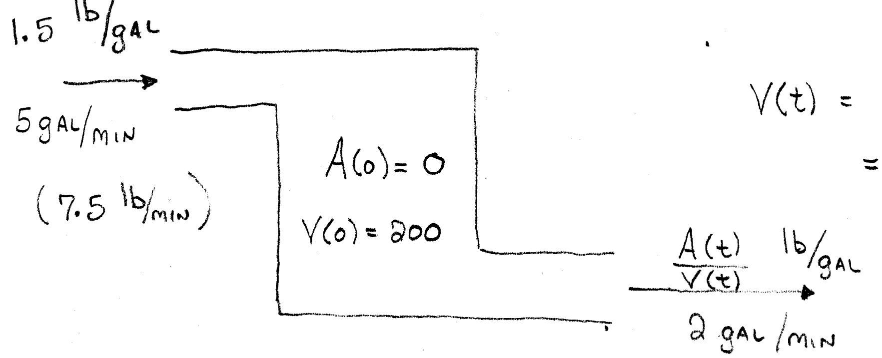
$\frac{\partial F}{\partial x} = y^3e^x + 4e^{2x}y \Rightarrow F(x,y) = y^3e^x + 2e^{2x}y + g(y)$

$\frac{\partial F}{\partial y} = 2e^{2x} + 3y^2e^x \Rightarrow F(x,y) = 2e^{2x}y + y^3e^x + h(x)$

$F(x,y) = 2e^{2x}y + y^3e^x$

$\boxed{\text{SOLUTION IS: } 2e^{2x}y + y^3e^x = C}$

9. (10 points) A 600-gallon tank is filled with 200 gallons of pure water. A spigot is opened above the tank, and a salt water solution containing 1.5 lb of salt per gallon begins flowing into the tank at a rate of 5 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 2 gal/min. What will be the concentration of salt in the solution at the precise moment when tank reaches its maximum capacity?



A(t) = AMOUNT OF SALT IN TANK AT TIME t (lbs)

V(t) = VOLUME OF TANK AT TIME t (GAL)

$= 200 + 3t$ (INCREASES AT 3 GAL/MIN)

TANK REACHES CAPACITY WHEN $200 + 3t = 600$

$t = \frac{400}{3}$

$$\frac{dA}{dt} = 7.5 - \frac{2A}{200+3t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{2}{200+3t}A = 7.5$$

$$\mu(t) = e^{\int \frac{2}{200+3t}dt} = e^{\frac{2}{3}\ln(200+3t)} = (200+3t)^{2/3}$$

$$(200+3t)^{2/3}A = \int 7.5(200+3t)^{2/3}dt = 7.5\left(\tfrac{3}{5}\right)\left(\tfrac{1}{3}\right)(200+3t)^{5/3} + C$$

$$A(t) = 1.5(200+3t) + \frac{C}{(200+3t)^{2/3}}$$

$$A(0) = 0 \Rightarrow 300 + \frac{C}{200^{2/3}} = 0 \Rightarrow C = -300(200)^{2/3} \approx 10259.86$$

$$A(t) = 1.5(200+3t) - \frac{10259.86}{(200+3t)^{2/3}}$$

$$\frac{A(400/3)}{600} \approx \frac{755.775\text{ lb}}{600\text{ gal}} \approx \boxed{1.26\text{ lb/gal}}$$