# Math 216 - Test 2a

March 14, 2012

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) According to Newton's Law of Cooling, the temperature $T$ at time $t$ of an object cooling in a medium of constant temperature $M$ is described by the differential equation

$$\frac{dT}{dt} = k(M - T),$$

where $k$ is some constant.

(a) Solve the differential equation.

$$\frac{1}{M-T}\, dT = k\, dt$$

$$-\ln|M-T| = kt + C_1$$

$$\ln|M-T| = C_2 - kt$$

$$|M-T| = e^{C_2 - kt} = C_3 e^{-kt}$$

$$M - T = C_4 e^{-kt}$$

$$\boxed{T(t) = M - Ce^{-kt}}$$

(b) An object at 120°F is moved into a large room with an ambient temperature of 72°F. The object cools to 100°F in 6 min. Use your result from part (a) to find a formula for the temperature of the object at time $t$.

$$T(t) = M - Ce^{-kt}$$

$$T(0) = 120$$

$$M = 72$$

$$T(6) = 100$$

$$120 = 72 - C \Rightarrow C = -48$$

$$T(t) = 72 + 48e^{-kt}$$

$$T(6) = 100 = 72 + 48e^{-6k}$$

$$\frac{28}{48} = e^{-6k} \Rightarrow k = \frac{\ln \frac{7}{12}}{-6}$$

$$\boxed{T(t) = 72 + 48e^{\frac{\ln \frac{7}{12}}{6} t}}$$

(c) When will the object reach 76°F?

$$76 = 72 + 48e^{\frac{\ln \frac{7}{12}}{6} t}$$

$$\frac{4}{48} = \frac{1}{12} = e^{\frac{\ln \frac{7}{12}}{6} t}$$

$$t = \frac{6 \ln \frac{1}{12}}{\ln \frac{7}{12}} \approx \boxed{27.66 \text{ MIN}}$$

2. (10 points) Solve: $y'' - y' - 30y = 0;\ y(0) = 2,\ y'(0) = -2$

CHAR. EQ: $r^2 - r - 30 = 0$

$(r-6)(r+5) = 0$

$r = 6,\ r = -5$

$y(x) = c_1 e^{6x} + c_2 e^{-5x}$

$y'(x) = 6c_1 e^{6x} - 5c_2 e^{-5x}$

$y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(0) = -2 \Rightarrow 6c_1 - 5c_2 = -2$

$5c_1 + 5c_2 = 10$

$6c_1 - 5c_2 = -2$

$11c_1 = 8$

$c_1 = \frac{8}{11}$

$c_2 = \frac{14}{11}$

$$\boxed{y(x) = \frac{8}{11} e^{6x} + \frac{14}{11} e^{-5x}}$$

3. (6 points) Solve: $4y'' + 20y' + 25y = 0$

CHAR. EQ.: $4r^2 + 20r + 5 = 0$

$(2r+5)(2r+5) = 0$

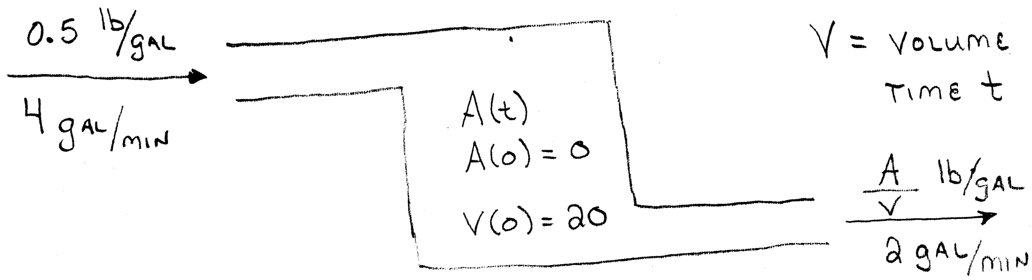
$r = -\frac{5}{2}$, MULTIPLICITY 2

$$\boxed{y(x) = c_1 e^{-\frac{5}{2}x} + c_2 x e^{-\frac{5}{2}x}}$$

4. (12 points) A 50-gal tank initially contains 20 gal of pure water. A saltwater solution containing 0.5 lb of salt for each gallon of water begins entering the tank at a rate of 4 gal/min. Simultaneously, a drain is opened at the bottom of the tank, allowing the solution to leave the tank at a rate of 2 gal/min. What is the salt content (in pounds) in the tank at the precise moment that the tank is full?

$A$ = AMOUNT OF SALT IN TANK AT TIME $t$

$V$ = VOLUME OF LIQUID AT TIME $t$ = $20 + 2t$




$$\frac{dA}{dt} = 2 - \frac{2A}{20+2t}$$

$$\frac{dA}{dt} = 2 - \frac{A}{10+t}$$

$$\frac{dA}{dt} + \frac{1}{10+t} A = 2$$

$$\mu(t) = e^{\int \frac{1}{10+t}\,dt} = e^{\ln|10+t|}$$

$$= |10+t| = 10+t \quad \text{SINCE } t \geq 0$$

$$(10+t)A = \int 2(10+t)\,dt$$

$$= 20t + t^2 + C$$

$A(0) = 0$

$\Rightarrow C = 0$

$$A(t) = \frac{20t+t^2}{10+t}$$

TANK IS FULL WHEN

$20 + 2t = 50$

$t = 15$

$A(15) = 21$

$\boxed{21 \text{ lbs}}$

5. (10 points) Consider the one-parameter family of curves described by

$$4x^2 + y^2 = C.$$

Find the family of orthogonal trajectories.

$$\frac{d}{dx}(4x^2+y^2) = 0$$

$$8x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-8x}{2y} = -\frac{4x}{y}.$$

ORTHO TRAJS SATISFY

$$\frac{dy}{dx} = \frac{y}{4x}$$

$$\frac{1}{y}\,dy = \frac{1}{4x}\,dx$$

$$\ln|y| = \frac{1}{4}\ln|x| + C_1$$

$$\ln|y| = \ln|x|^{1/4} + C_1$$

$$|y| = e^{\ln|x|^{1/4} + C_1} = C_2|x|^{1/4}$$

$$y = C_3|x|^{1/4} \quad \text{OR} \quad \boxed{y = C_3\sqrt[4]{|x|}}$$

# Math 216 - Test 2b
March 14, 2012

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) An object is launched upward so that its velocity (in m/s) at time $t$ is described by the initial value problem

$$\frac{dv}{dt} = -19.6 - \frac{v}{20}, \quad v(0) = 200.$$

(a) Solve the initial value problem.

$$v' + \frac{1}{20}v = -19.6$$

$$\mu(t) = e^{\int \frac{1}{20}dt} = e^{\frac{1}{20}t}$$

$$e^{\frac{1}{20}t} v = \int -19.6 e^{\frac{1}{20}t} dt$$

$$= -392 e^{\frac{1}{20}t} + C$$

$$v(t) = -392 + Ce^{-\frac{1}{20}t}$$

$$v(0) = 200 = -392 + C$$

$$C = 592$$

$$\boxed{v(t) = -392 + 592e^{-\frac{1}{20}t}}$$

(b) If the initial height of the object is 30 m, find a formula for the height at time $t$.

$$h(t) = -392t - 11840e^{-\frac{1}{20}t} + C$$

$$h(0) = 30 \Rightarrow -11840 + C = 30$$

$$C = 11870$$

$$\boxed{h(t) = -392t - 11840e^{-\frac{1}{20}t} + 11870}$$

(c) Find the object's maximum height.

$v(t) = 0$ WHEN

$$e^{-\frac{1}{20}t} = \frac{392}{592}$$

$$t = -20\ln\left(\frac{392}{592}\right)$$

$t \approx 8.245$ SECONDS

$$\boxed{h\left(-20\ln\left(\frac{392}{592}\right)\right) \approx 798.001 \text{ m}}$$

2. (10 points) Determine the recursive formulas for the Taylor method of order 3 for the IVP

$$\frac{dy}{dx} = xe^y, \quad y(0) = 1.$$

Then use $h = 0.1$ to approximate $y(0.2)$.

$$f(x,y) = xe^y$$

$$f'(x,y) = xe^y \frac{dy}{dx} + e^y = x^2 e^{2y} + e^y$$

$$f''(x,y) = 2x^2 e^{2y} \frac{dy}{dx} + 2xe^{2y} + e^y \frac{dy}{dx}$$

$$= 2x^3 e^{3y} + 3xe^{2y}$$

$$\boxed{\begin{aligned} y_{N+1} &= y_N + h\left(x_N e^{y_N}\right) + \frac{h^2}{2}\left(x_N^2 e^{2y_N} + e^{y_N}\right) + \frac{h^3}{6}\left(2x_N^3 e^{3y_N} + 3x_N e^{2y_N}\right) \\ x_{N+1} &= x_N + h \end{aligned}}$$

$y_0 = 1$ $\quad h = 0.1$

$x_0 = 0$

$$y_1 = 1 + 0.1(0) + \frac{0.1^2}{2}(e) + \frac{0.1^3}{6}(0) = 1.01359\ldots$$

$x_1 = 0.1$

$$y_2 = 1.01359 + 0.1(0.2755\ldots) + \frac{0.1^2}{2}(2.8314\ldots) + \frac{0.1^3}{6}(2.3196\ldots)$$

$$\approx 1.05569$$

$x_2 = 0.2$

$$\boxed{y(0.2) \approx 1.05569}$$

3. (10 points) Solve: $y^{(4)} - 3y'' - 4y = 0$; $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$, $y'''(0) = 4$

Char eq: $r^4 - 3r^2 - 4 = 0$

$$(r^2-4)(r^2+1) = 0$$

$$(r-2)(r+2)(r^2+1) = 0$$

$$r = 2,\ r = -2,\ r = i,\ r = -i$$

$$\{ e^{2x},\ e^{-2x},\ \cos x,\ \sin x \}$$

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x$$

$$y'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x} - c_3 \sin x + c_4 \cos x$$

$$y''(x) = 4c_1 e^{2x} + 4c_2 e^{-2x} - c_3 \cos x - c_4 \sin x$$

$$y'''(x) = 8c_1 e^{2x} - 8c_2 e^{-2x} + c_3 \sin x - c_4 \cos x$$

IC's give

$$c_1 + c_2 + c_3 = 1 \qquad 4c_1 + 4c_2 - c_3 = 3$$

$$2c_1 - 2c_2 + c_4 = 2 \qquad 8c_1 - 8c_2 - c_4 = 4$$

$$\Rightarrow \quad c_1 = \frac{7}{10},\quad c_2 = \frac{1}{10},\quad c_3 = \frac{2}{10},\quad c_4 = \frac{8}{10}$$

$$\boxed{y(x) = \frac{7}{10} e^{2x} + \frac{1}{10} e^{-2x} + \frac{2}{10} \cos x + \frac{8}{10} \sin x}$$

4. (5 points) Solve: $y'' - 2y' + 2y = 0$

Char eq: $r^2 - 2r + 2 = 0$

$$r^2 - 2r + 1 = -1$$

$$(r-1)^2 = -1$$

$$r - 1 = \pm i$$

$$r = 1 \pm i$$

$$\boxed{y(x) = c_1 e^{x} \cos x + c_2 e^{x} \sin x}$$

5. (10 points) Consider the initial value problem

$$\frac{dy}{dx} = -20y, \quad y(0) = 1.$$

(a) Solve the IVP.

$y' + 20y = 0$

CHAR EQ: $r + 20 = 0$

$r = -20$

$y(x) = c_1 e^{-20x}$

$y(0) = 1 \Rightarrow c_1 = 1$

$\boxed{y(x) = e^{-20x}}$

(b) Use the improved Euler's method with $h = 0.1$ to approximate $y(1)$. Be sure to look at your intermediate results.

$y_0 = 1$
$y_1 = 1$
$y_2 = 1$
$\vdots$
$y_{10} = 1$

$y(1) \approx 1$

AND ALL INTERMEDIATE RESULTS ARE ALSO 1.

(c) Compare your approximation with the exact value of $y(1)$.

$y(1) = e^{-20} \approx 2.06 \times 10^{-9}$

THE APPROXIMATION VIA IMPROVED EULER'S METHOD IS NOT EVEN CLOSE!

(d) Use the improved Euler's method with $h = 0.01$ to approximate $y(1)$.

USING $h = 0.01$, WE GET

$y(1) \approx 2.406 \times 10^{-9}$

AT LEAST IT'S KINDA CLOSE.

(e) Explain why you got the results you did in part (b).

TAKE A LOOK AT THE 1ST STEP.

$y_1^* = 1 + 0.1(-20) = -1$

$\underbrace{y_1 = 1 + \frac{0.1}{2}(-20 + 20) = 1}_{\text{SAME THING WILL CONTINUE TO HAPPEN}}$

BECAUSE IMPROVED EULER COINCIDENTALLY (SOMEWHAT) GENERATED $y_0 = y_1 = 1$ AND $f(x,y) = -20y$ DOES NOT DEPEND ON X, ALL APPROXIMATIONS WILL BE $y_N = 1$.

6. (5 points) The solution of the initial value problem

$$\frac{dy}{dx} = y^2 - 2e^x y + e^{2x} + e^x, \quad y(0) = 3$$

has a vertical asymptote at a point in the interval $[0, 2]$. By experimenting with the classic 4th-order Runge-Kutta method, approximate this point.

SEE THE ATTACHED OUTPUT FROM RK4

USING $h = 0.001$.

IT LOOKS LIKE THE

VERTICAL ASYMPTOTE IS

VERY CLOSE TO $X = 0.5$.

$x488 = 0.488$ , $y488 = 84.96233293265611$

$x489 = 0.489$ , $y489 = 92.53969041907769$

$x490 = 0.49$ , $y490 = 101.6321793128971$

$x491 = 0.491$ , $y491 = 112.7448293373258$

$x492 = 0.492$ , $y492 = 126.6351694085914$

$x493 = 0.493$ , $y493 = 144.4935597550258$

$x494 = 0.494$ , $y494 = 168.303804004754$

$x495 = 0.495$ , $y495 = 201.6362765578846$

$x496 = 0.496$ , $y496 = 251.6295725039352$

$x497 = 0.497$ , $y497 = 334.9266764520596$

$x498 = 0.498$ , $y498 = 501.3048268975458$

$x499 = 0.499$ , $y499 = 994.5461896403247$

$x500 = 0.5$ , $y500 = 8200.750523804138$

$x501 = 0.501$ , $y501 = 1.0100511850659802\ 10^{14}$

$x502 = 0.502$ , $y502 = 4.7751049598733946\ 10^{174}$

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EXPT: floating point overflow.
 -- an error. To debug this try: debugmode(true);
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