Math 216 - Test 3
April 25, 2012

Name key
Score ____________

Show all work. Supply explanations when necessary.
YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. (10 points) One solution of the equation

$$(1-x^2)y'' + 2xy' - 2y = 0$$

is $y_1(x) = x$. Find the general solution.

$$y'' + \frac{2x}{1-x^2}y' - \frac{2}{1-x^2}y = 0$$

$$\int \frac{2x}{1-x^2}\,dx = -\ln|1-x^2|$$

$$v(x) = \int \frac{1}{x^2} e^{\ln|1-x^2|}\,dx = \int \frac{1-x^2}{x^2}\,dx = \int (x^{-2} - 1)\,dx = -\frac{1}{x} - x$$

$$y_2(x) = v(x)y_1(x) = \left(-\frac{1}{x} - x\right)x = \left(-1-x^2\right) = (-1)(1+x^2)$$

$$\boxed{\text{GENERAL SOLUTION IS } \quad y(x) = c_1 x + c_2(1+x^2)}$$

2. (20 points) Solve the following initial value problem.

$$xy'' + y' = x; \quad y(1) = 1,\ y'(1) = -1/2$$

CAUCHY-EULER: $x^2y'' + xy' = x^2$

Homo eq: $x^2y''+xy'=0$

ASSUME $x>0$ AND LET $x=e^t$.

THIS GIVES

$$y''(t) = 0 \Rightarrow y(t) = c_1 + c_2 t$$

OR

$$y(x) = c_1 + c_2 \ln x$$

NONHOMO EQ:

$y'' + \frac{1}{x}y' = 1$

$g(x) = 1$

$y_1(x) = 1$

$y_2(x) = \ln x$

$$W(x) = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$$

$$v_1(x) = \int -x \ln x \, dx = -\frac{x^2}{2}\ln x + \int \frac{x}{2}dx$$

$u = \ln x$

$dv = -x\,dx$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$v_2(x) = \int x \, dx = \frac{x^2}{2}$$

$$y_p(x) = -\frac{x^2}{2}\ln x + \frac{x^2}{4} + \frac{x^2}{2}\ln x = \frac{x^2}{4}$$

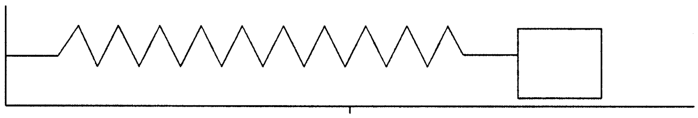
$$y(x) = c_1 + c_2 \ln x + \frac{x^2}{4}$$

$y(1) = 1 \Rightarrow c_1 = \frac{3}{4}$

$$\boxed{y(x) = \frac{3}{4} - \ln x + \frac{x^2}{4}}$$

$y'(x) = \frac{c_2}{x} + \frac{x}{2}, \quad y'(1) = -\frac{1}{2} \Rightarrow c_2 = -1$

3. (10 points) A 1/2-kg mass is attached to a spring with spring constant 5 N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 2 m to the right of equilibrium (stretching the spring) and released from rest. Is the system underdamped, overdamped, or critically damped? How do you know? Find the equation of motion. Write your final result in terms of a single trig function with phase shift.



$x = 0$
(Equilibrium)

$\frac{1}{2}x'' + 5x + x' = 0; \quad x(0) = 2, \quad x'(0) = 0$

$x'' + 2x' + 10x = 0; \quad x(0) = 2, \quad x'(0) = 0$

$b^2 - 4mk = 1^2 - 4\left(\frac{1}{2}\right)(5) = -9 \Rightarrow$ SYSTEM IS UNDERDAMPED.

CHAR eq: $r^2 + 2r + 10 = 0$

$r^2 + 2r + 1 = -9$

$(r+1)^2 = -9$

$r = -1 \pm \sqrt{-9}$

$\alpha = -1, \ \beta = 3$

$x(t) = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right)$

$x(0) = 2 \Rightarrow c_1 = 2$

$x'(t) = -e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right) + e^{-t}\left(-3c_1 \sin 3t + 3c_2 \cos 3t\right)$

$x'(0) = 0 \Rightarrow -c_1 + 3c_2 = 0$

$\Rightarrow 3c_2 = 2 \Rightarrow c_2 = \frac{2}{3}$

$x(t) = e^{-t}\left(2\cos 3t + \frac{2}{3}\sin 3t\right)$

$A = \sqrt{4 + \frac{4}{9}} = \sqrt{\frac{40}{9}}$

c_1 & c_2 ARE POSITIVE

$\Rightarrow \varphi$ IS A 1^{ST} QUAD ANGLE

$\tan \varphi = 2/\tfrac{2}{3} = 3 \Rightarrow \varphi = \tan^{-1} 3$

$$\boxed{x(t) = \frac{\sqrt{40}}{3} e^{-t} \sin\left(3t + \tan^{-1} 3\right)}$$

4. (20 points) Solve the following initial value problem.

$$y'' - 10y' + 25y = 5x^2e^{5x}; \quad y(0) = 1,\ y'(0) = 2$$

Homo eq: $y'' - 10y' + 25y = 0$

$r^2 - 10r + 25 = (r-5)^2 = 0 \Rightarrow r = 5$

$y_h(x) = c_1 e^{5x} + c_2 x e^{5x}$

NonHomo eq: $g(x) = 5x^2 e^{5x}$

$y_p(x) = x^s(Ax^2 + Bx + C)e^{5x}$

Must choose $s = 2$

$y_p(x) = (Ax^4 + Bx^3 + Cx^2)e^{5x}$

$y_p'(x) = 5(Ax^4 + Bx^3 + Cx^2)e^{5x} + (4Ax^3 + 3Bx^2 + 2Cx)e^{5x}$

$y_p''(x) = 25(Ax^4 + Bx^3 + Cx^2)e^{5x} + 5(4Ax^3 + 3Bx^2 + 2Cx)e^{5x} + 5(4Ax^3 + 3Bx^2 + 2Cx)e^{5x} + (12Ax^2 + 6Bx + 2C)e^{5x}$

$y'' - 10y' + 25y = 5x^2e^{5x}$

$\Downarrow$

$x^4: 25A - 50A + 25A = 0$

$x^3: 25B + 20A + 20A - 50B - 40A + 25B = 0$

$x^2: 25C + 15B + 15B + 12A - 50C - 30B + 25C = 5$

$x: 10C + 10C + 6B - 20C = 0$

Const: $2C = 0$

$12A = 5$

$6B = 0 \quad \Rightarrow \quad A = \frac{5}{12},\ B = 0,\ C = 0$

$2C = 0$

$y_p(x) = \frac{5}{12}x^4e^{5x}$

$y(x) = c_1e^{5x} + c_2xe^{5x} + \frac{5}{12}x^4e^{5x}$

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(x) = 5c_1e^{5x} + c_2e^{5x} + 5c_2xe^{5x} + \frac{5}{3}x^3e^{5x} + \frac{25}{12}x^4e^{5x}$

$y'(0) = 2 \Rightarrow 5c_1 + c_2 = 2$

$5 + c_2 = 2$

$\Rightarrow c_2 = -3$

$$\boxed{y(x) = e^{5x} - 3xe^{5x} + \frac{5}{12}x^4e^{5x}}$$

5. (20 points) Find the general solution: $y'' + 3y' + 2y = \sin(e^x)$

Homo eq: $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0 \Rightarrow r = -2,\ r = -1$

$y_h(x) = c_1 e^{-2x} + c_2 e^{-x}$

NonHomo eq: $g(x) = \sin(e^x)$

$y_1(x) = e^{-2x}, \quad y_2(x) = e^{-x}$

$$W(x) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}$$

$$V_1(x) = \int \frac{-e^{-x}\sin(e^x)}{e^{-3x}}\,dx = \int -e^{2x}\sin(e^x)\,dx$$

$u = e^x$

$du = e^x\,dx$

$\Rightarrow -\int u\sin u\,du =$

$-$	u	$\sin u$
$+$	1	$-\cos u$
$-$	0	$-\sin u$

$= u\cos u - \sin u$

$= e^x\cos e^x - \sin e^x$

$$V_2(x) = \int \frac{e^{-2x}\sin(e^x)}{e^{-3x}}\,dx = \int e^x\sin(e^x)\,dx$$

$u = e^x$

$du = e^x\,dx$

$\Rightarrow \int \sin u\,du$

$= -\cos u$

$= -\cos e^x$

$y_p(x) = e^{-2x}(e^x\cos e^x - \sin e^x) + e^{-x}(-\cos e^x)$

$= -e^{-2x}\sin e^x$

$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x}\sin(e^x)}$$

6. (20 points) Solve the following initial value problem.

$$\frac{dx}{dt} = 2x + y - e^{2t}; \quad x(0) = 1,$$

$$\frac{dy}{dt} = x + 2y; \quad y(0) = -1$$

$$(D-2)x - y = -e^{2t}$$
$$(D-2)y - x = 0$$

$$\Rightarrow \quad (D-2)x - y = -e^{2t}$$
$$-(D-2)x + (D^2-4D+4)y = 0$$

$$(D^2-4D+3)y = -e^{2t}$$

$$y'' - 4y' + 3y = -e^{2t}$$

Homo eq: $y'' - 4y' + 3y = 0$

$$r^2 - 4r + 3 = (r-1)(r-3) = 0$$

$$r = 1, \; r = 3$$

$$y_h(t) = c_1 e^{t} + c_2 e^{3t}$$

Non Homo eq: $g(t) = -e^{2t}$

$$y_p(t) = Ae^{2t}$$

$$y_p'(t) = 2Ae^{2t} \qquad y_p''(t) = 4Ae^{2t}$$

$$(4A - 8A + 3A)e^{2t} = -e^{2t}$$

$$\Downarrow$$

$$A = 1$$

$$y(t) = c_1 e^{t} + c_2 e^{3t} + e^{2t}$$

$$y'(t) = c_1 e^{t} + 3c_2 e^{3t} + 2e^{2t}$$

$$x(t) = y'(t) - 2y(t)$$

$$x(t) = -c_1 e^{t} + c_2 e^{3t}$$

$$x(0) = 1 \Rightarrow \quad -c_1 + c_2 = 1$$

$$y(0) = -1 \Rightarrow \quad c_1 + c_2 + 1 = -1$$

$$2c_2 + 1 = 0$$

$$c_2 = -\frac{1}{2}$$

$$c_1 = -\frac{3}{2}$$

$$\boxed{\begin{aligned} x(t) &= \frac{3}{2}e^{t} - \frac{1}{2}e^{3t} \\ y(t) &= -\frac{3}{2}e^{t} - \frac{1}{2}e^{3t} + e^{2t} \end{aligned}}$$