Math 216 - 1st Final Exam

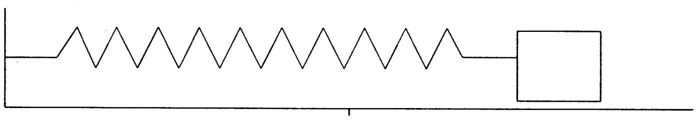
May 2, 2012

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) A 3-kg mass is attached to a spring with spring constant 3 N/m. The damping constant for the system is 6 N-sec/m. The mass is moved 1 m to the LEFT of equilibrium (compressing the spring) and pushed to the RIGHT at 1 m/sec. Set up and solve the initial value problem that describes the displacement of the mass from equilibrium. Is the mass-spring system underdamped, overdamped, or critically damped?



$x = 0$
(Equilibrium)

$3x'' + 6x' + 3x = 0, \quad x(0) = -1, \quad x'(0) = 1$

$b^2 - 4mk = 36 - 4(3)(3) = 0 \Rightarrow$ System is critically damped

$3x'' + 6x' + 3x = 0 \Rightarrow x'' + 2x' + x = 0$

Char eq: $r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow r = -1$ (twice)

$x(t) = c_1 e^{-t} + c_2 t e^{-t}$

$x(0) = -1 \Rightarrow c_1 = -1$

$x'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$x'(0) = 1 \Rightarrow -c_1 + c_2 = 1 \Rightarrow c_2 = 0$

$\boxed{x(t) = -e^{-t}}$

2. (10 points) Find the general solution of $y''' + 2y'' + 2y' = 0$.

CHAR eq IS $r^3 + 2r^2 + 2r = 0$

$$r(r^2 + 2r + 2) = 0$$

$$r = 0, \quad r = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

SOLUTION SET

$$\{ 1, e^{-x}\cos x, e^{-x}\sin x \}$$

$$\boxed{y(x) = C_1 + C_2 e^{-x}\cos x + C_3 e^{-x}\sin x}$$

3. (15 points) Solve: $xy' + 2y = x^2, \quad y(1) = 1$

$$y' + \frac{2}{x}y = x$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$$

$$x^2 y(x) = \int x^3 dx \Rightarrow x^2 y(x) = \frac{1}{4}x^4 + C$$

$$y(x) = \frac{1}{4}x^2 + \frac{C}{x^2}$$

$$y(1) = 1 \Rightarrow \frac{1}{4} + C = 1$$

$$C = \frac{3}{4}$$

$$\boxed{y(x) = \frac{x^2}{4} + \frac{3}{4x^2}}$$

4. (15 points) One solution of the following equation is $y(x) = \sqrt{x}$.

$$4x^2y'' - (20x^2 + 4x)y' + (10x + 3)y = 0$$

We're assuming $x > 0$

Find the general solution. (Hint: Be sure to rewrite the equation in standard form.)

$$y'' - \left(5 + \frac{1}{x}\right)y' + \left(\frac{10x+3}{4x^2}\right)y = 0$$

$$\int \left(-5 - \frac{1}{x}\right) dx = -5x - \ln x$$

$$v(x) = \int \frac{1}{x} e^{5x + \ln x} dx = \int e^{5x} dx = \frac{1}{5} e^{5x}$$

$$y_2(x) = \frac{1}{5}\sqrt{x}\, e^{5x} \Rightarrow \boxed{y(x) = c_1\sqrt{x} + c_2\sqrt{x}\, e^{5x}}$$

5. (12 points) Solve: $\quad y' = xy^{-3}e^x, \quad y(0) = 1$

$$y^3 dy = xe^x dx$$

$$\int y^3 dy = \int xe^x dx$$

+	x	e^x
−	1	e^x
+	0	e^x

$$\frac{1}{4}y^4 = xe^x - e^x + C$$

$$y(0) = 1 \Rightarrow \frac{1}{4} = -1 + C \Rightarrow C = \frac{5}{4}$$

$$\frac{1}{4}y^4 = xe^x - e^x + \frac{5}{4}$$

$$y^4 = 4xe^x - 4e^x + 5$$

$$\boxed{y(x) = \left(4xe^x - 4e^x + 5\right)^{1/4}}$$

6. (12 points) Show that the following equation is exact and solve.

$$\underbrace{(4xy^{1/2} + x^2 - 2x^{-1/2}y^{1/2})}_{M(x,y)}\,dx + \underbrace{(2y + x^2y^{-1/2} - 2x^{1/2}y^{-1/2})}_{N(x,y)}\,dy = 0$$

$$\underbrace{\frac{\partial M}{\partial y} = 2xy^{-1/2} - x^{-1/2}y^{-1/2} = \frac{\partial N}{\partial x} = 2xy^{-1/2} - x^{-1/2}y^{-1/2}}_{\text{EXACT !}}$$

$$\frac{\partial F}{\partial x} = 4xy^{1/2} + x^2 - 2x^{-1/2}y^{1/2} \Rightarrow F(x,y) = 2x^2y^{1/2} + \frac{1}{3}x^3 - 4x^{1/2}y^{1/2} + g(y)$$

$$\frac{\partial F}{\partial y} = 2y + x^2y^{-1/2} - 2x^{1/2}y^{-1/2} \Rightarrow F(x,y) = y^2 + 2x^2y^{1/2} - 4x^{1/2}y^{1/2} + h(x)$$

$$F(x,y) = 2x^2y^{1/2} - 4x^{1/2}y^{1/2} + y^2 + \frac{1}{3}x^3$$

SOLUTION IS $2x^2\sqrt{y} - 4\sqrt{xy} + y^2 + \frac{1}{3}x^3 = C$

7. (10 points) Use Euler's method with a step size of $h = 0.5$ to approximate $y(3)$, where $y(x)$ is the solution of the initial value problem $y' = xy^2$, $y(2) = 1$.

$$y_{N+1} = y_N + h f(x_N, y_N), \quad f(x,y) = xy^2$$

$$x_{N+1} = x_N + h$$

$y_0 = 1$

$x_0 = 2$

$y_1 = 1 + 0.5(2)(1)^2 = 2$

$x_1 = 2.5$

$y_2 = 2 + 0.5(2.5)(2)^2 = 7$

$x_2 = 3$

$\boxed{y(3) \approx 7}$

8. (20 points) Solve.

$$\begin{aligned} y' &- 2y + z = 0, & y(0) &= 1 \\ z' &- y - 2z = 0, & z(0) &= 0 \end{aligned}$$

$$(D-2)y + z = 0$$

$$-y + (D-2)z = 0.$$

$$(D-2)y + z = 0$$

$$-(D-2)y + (D-2)^2 z = (D-2)0$$

$$(D^2-4D+5)z = 0.$$

$$z'' - 4z' + 5z = 0$$

$$r^2 - 4r + 5 = 0$$

$$r = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\alpha = 2, \beta = 1$$

$$z(t) = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t$$

$$y = z' - 2z \Rightarrow$$

$$y(t) = -c_1 e^{2t}\sin t + 2c_1 e^{2t}\cos t$$

$$+ c_2 e^{2t}\cos t + 2c_2 e^{2t}\sin t$$

$$- 2c_1 e^{2t}\cos t - 2c_2 e^{2t}\sin t$$

$$y(t) = -c_1 e^{2t}\sin t + c_2 e^{t}\cos t$$

$$y(0) = 1 \Rightarrow c_2 = 1$$

$$z(0) = 0 \Rightarrow c_1 = 0$$

$$\boxed{\begin{aligned} y(t) &= e^{2t}\cos t \\ z(t) &= e^{2t}\sin t \end{aligned}}$$

9. (20 points) Use undetermined coefficients to find the general solution of the following equation:

$$x'' - 4x' + 4x = 2 + 3e^t$$

Homo eq: $x'' - 4x' + 4x = 0$

CHAR eq: $r^2 - 4r + 4 = (r-2)^2 = 0$

$r = 2$ (TWICE)

$X_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

NON HOMO eq: $g(x) = 2 + 3e^t$

$\Rightarrow X_p(t) = A + Be^t$

$X_p'(t) = Be^t$

$X_p''(t) = Be^t$

$Be^t - 4Be^t + 4A + 4Be^t = 2 + 3e^t$

$Be^t + 4A = 2 + 3e^t$

$A = \frac{1}{2}, \quad B = 3$

$X_p(t) = \frac{1}{2} + 3e^t$

$X(t) = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{2} + 3e^t$

10. (10 points) Find the orthogonal trajectories for the family of curves described by the equation $Cy^2 = x^3$.

$C\left(2y \frac{dy}{dx}\right) = 3x^2$

$\frac{x^3}{y^2}\left(2y \frac{dy}{dx}\right) = 3x^2$

$\frac{2x^3}{y} \frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2 y}{2x^3} = \frac{3y}{2x}$

ORTHO Trajs SATISFY

$\frac{dy}{dx} = \frac{-2x}{3y}$

$3y\,dx = -2x\,dx$

$\frac{3}{2}y^2 = -x^2 + C$

$\boxed{\frac{3}{2}y^2 + x^2 = C}$

11. (6 points) For $x > 0$, let $y_1(x) = \ln x^5$ and $y_2(x) = \ln x$. Compute the Wronskian of y_1 and y_2. Briefly explain why $y(x) = c_1 y_1(x) + c_2 y_2(x)$ cannot be the general solution of a 2nd-order, linear, homogeneous differential equation.

$5 \ln x$

$$W(x) = \begin{vmatrix} \ln x^5 & \ln x \\ \frac{5}{x} & \frac{1}{x} \end{vmatrix} = \frac{\ln x^5}{x} - \frac{5 \ln x}{x} = 0$$

$y_1(x)$ AND $y_2(x)$ ARE NOT LINEARLY INDEPENDENT.

A GENERAL SOLUTION MUST BE A LINEAR COMBINATION OF TWO LINEARLY INDEPENDENT FUNCS.

12. (4 points) What does it mean for two families of curves to be orthogonal trajectories of one another?

AT EVERY POINT WHERE A MEMBER OF ONE FAMILY INTERSECTS A MEMBER OF THE OTHER FAMILY, THE TANGENT LINES ARE PERPENDICULAR.

13. (4 points) Write a differential equation that would describe a mass-spring system that is overdamped. Do not solve your equation, but explain how you know the system is overdamped.

OVERDAMPED $\Rightarrow$ $b^2 - 4mk > 0$

CHOOSE $b = 5$, $m = 1$, $k = 1$

$$x'' + 5x' + x = 0$$