Math 216 - 2nd Final Exam

May 14, 2012

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Solve: $y'' + 5y' + 6y = 0, \quad y(0) = y'(0) = 1$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, \ r = -3$

$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(x) = -2c_1 e^{-2x} - 3c_2 e^{-3x}$

$y'(0) = 1 \Rightarrow -2c_1 - 3c_2 = 1$

$$\begin{array}{r} 2c_1 + 2c_2 = 2 \\ -2c_1 - 3c_2 = 1 \\ \hline -c_2 = 3 \end{array} \Rightarrow c_2 = -3$$

$\Rightarrow c_1 = 4$

$$\boxed{y(x) = 4e^{-2x} - 3e^{-3x}}$$

2. (13 points) Use Euler's method with $h = 0.1$ to approximate $y(0.2)$ for the initial value problem. Then find the exact solution.

$$y' + 5y = 3e^x, \quad y(0) = 1$$

$f(x,y) = 3e^x - 5y$

$y_0 = 1$

$x_0 = 0$

$y_1 = 1 + 0.1\left(3e^0 - 5(1)\right)$

$= 1 + 0.1(-2) = 0.8$

$x_1 = 0.1$

$y_2 = 0.8 + 0.1\left(3e^{0.1} - 5(0.8)\right)$

$= 0.8 + 0.1(-0.684487\ldots)$

≈ 0.7315513

$x_2 = 0.2$

$$\boxed{y(0.2) \approx 0.7315513}$$

$\mu(x) = e^{\int 5\,dx} = e^{5x}$

$e^{5x} y = \int e^{5x} \cdot 3e^x \, dx$

$e^{5x} y = \frac{1}{2} e^{6x} + C$

$y = \frac{1}{2} e^x + Ce^{-5x}$

$y(0) = 1 \Rightarrow \frac{1}{2} + C = 1 \Rightarrow C = \frac{1}{2}$

$$\boxed{\begin{array}{l} y(x) = \frac{1}{2} e^x + \frac{1}{2} e^{-5x} \\ y(0.2) \approx 0.79464 \end{array}}$$

3. (15 points) Use variation of parameters to solve the following differential equation.

$$x'' - 4x' + 4x = \frac{e^{2t}}{t^2}, \quad t > 0$$

Homo eq: $x'' - 4x' + 4x = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2, 2$

$x_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

NonHomo eq: $g(t) = \dfrac{e^{2t}}{t^2}$

$x_1(t) = e^{2t}, \quad x_2(t) = t e^{2t}$

$$W(t) = \begin{vmatrix} e^{2t} & t e^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{vmatrix}$$

$$= e^{4t}$$

$$V_1(t) = \int -\frac{e^{2t} t e^{2t}}{t^2 e^{4t}} \, dt = \int -\frac{1}{t} \, dt$$

$$= -\ln t$$

$$V_2(t) = \int \frac{e^{2t} e^{2t}}{t^2 e^{4t}} \, dt = \int \frac{1}{t^2} \, dt = -\frac{1}{t}$$

$x_p(t) = -e^{2t} \ln t - e^{2t}$

$x(t) = x_h(t) + x_p(t)$

$$x(t) = c_3 e^{2t} + c_2 t e^{2t} - e^{2t} \ln t$$

4. (20 points) Use any method to solve the following system of equations.

$$\begin{aligned} x' &= 3x - y - 1, & x(0) &= 0 \\ y' &= x + y + 4e^t, & y(0) &= -2 \end{aligned}$$

$$(D-3)x + y = -1$$

$$-x + (D-1)y = 4e^t \quad \leftarrow \text{"MULT" BY } D-3$$

$$(D-3)x + y = -1$$

$$-(D-3)x + (D^2-4D+3)y = -8e^t$$

$$(D^2-4D+4)y = -8e^t - 1$$

$$y'' - 4y' + 4y = -8e^t - 1$$

Homo eq: $y'' - 4y' + 4y = 0$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0; \; r = 2, 2$$

$$y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$$

NonHomo Eq: $g(t) = -8e^t - 1$

$$y_p(t) = Ae^t + B$$

$$y_p'(t) = Ae^t$$

$$y_p''(t) = Ae^t$$

$$Ae^t - 4Ae^t + 4Ae^t + 4B = -8e^t - 1$$

$$A = -8$$

$$B = -\frac{1}{4}$$

$$y_p(t) = -8e^t - \frac{1}{4}$$

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} - 8e^t - \frac{1}{4}$$

$$\begin{aligned} X(t) &= y' - y - 4e^t \\ &= 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} - 8e^t \\ &\quad - c_1 e^{2t} - c_2 t e^{2t} + 8e^t + \frac{1}{4} \\ &\quad - 4e^t \end{aligned}$$

$$X(t) = (c_1 + c_2)e^{2t} + c_2 t e^{2t} - 4e^t + \frac{1}{4}$$

$$X(0) = 0 \Rightarrow c_1 + c_2 - 4 + \frac{1}{4} = 0$$

$$y(0) = -2 \Rightarrow c_1 - 8 - \frac{1}{4} = -2$$

$$c_1 = 25/4$$

$$c_2 = -5/2$$

$$X(t) = \frac{15}{4}e^{2t} - \frac{5}{2}te^{2t} - 4e^t + \frac{1}{4}$$

$$y(t) = \frac{25}{4}e^{2t} - \frac{5}{2}te^{2t} - 8e^t - \frac{1}{4}$$

5. (16 points) Use Laplace transform techniques to solve the following initial value problem. You may use a computer algebra system for any partial fraction decompositions.

$$y'' - 2y' + y = -16e^{-3t}; \quad y(0) = 1,\ y'(0) = 8$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = -16\,\mathcal{L}\{e^{-3t}\}$$

LET $Y(s) = \mathcal{L}\{y\}$.

$$(s^2 Y - s(1) - 8) - 2(sY - 1) + Y = \frac{-16}{s+3}$$

$$(s^2 - 2s + 1)Y - s - 6 = \frac{-16}{s+3}$$

$$Y = \frac{\frac{-16}{s+3} + s + 6}{s^2 - 2s + 1} = \underbrace{\frac{-1}{s+3} + \frac{2}{s-1}}_{\text{PFD}} + \frac{3}{(s-1)^2}$$

$$Y(s) = \frac{-1}{s+3} + \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\Rightarrow \boxed{y(t) = -e^{-3t} + 2e^{t} + 3te^{t}}$$

6. (10 points) Solve.

$$\underbrace{(2xe^{3y} + e^x)}_{M}\, dx + \underbrace{(3x^2e^{3y} - y^2)}_{N}\, dy = 0$$

$$\frac{\partial M}{\partial y} = 6xe^{3y} = \frac{\partial N}{\partial x} = 6xe^{3y} \Rightarrow$$ EQUATION IS EXACT.

$$\frac{\partial F}{\partial x} = 2xe^{3y} + e^x \Rightarrow F(x,y) = x^2e^{3y} + e^x + g(y)$$

$$\frac{\partial F}{\partial y} = 3x^2e^{3y} - y^2 \Rightarrow F(x,y) = x^2e^{3y} - \frac{1}{3}y^3 + h(x)$$

$$\boxed{x^2e^{3y} - \frac{1}{3}y^3 + e^x = C}$$

7. (10 points) Solve.

$$y' = y^2 \sin x, \quad y(2\pi) = 1$$

$$\frac{dy}{y^2} = \sin x\, dx$$

$$\int \frac{1}{y^2}\, dy = \int \sin x\, dx$$

$$-\frac{1}{y} = -\cos x + C_1$$

$$\frac{1}{y} = \cos x + C_2$$

$$y(2\pi) = 1 \Rightarrow 1 = \cos 2\pi + C_2$$

$$\Rightarrow C_2 = 0$$

$$\frac{1}{y} = \cos x \Rightarrow \boxed{y(x) = \sec x}$$

8. (16 points) Consider the equation $(x^2+1)y'' - 2xy' + 2y = 0$.

(a) Verify that $y_1(x) = (1-x^2)$ and $y_2(x) = x$ are solutions. Do not solve; simply verify.

$y_1(x) = 1 - x^2$

$y_1'(x) = -2x$

$y_1''(x) = -2$

$(x^2+1)(-2) - 2x(-2x) + 2(1-x^2)$

$= -2x^2 - 2 + 4x^2 + 2 - 2x^2 = 0$ ✓

$y_2(x) = x$

$y_2'(x) = 1$

$y_2''(x) = 0$

$(x^2+1)(0) - 2x + 2x$

$= -2x + 2x = 0$ ✓

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent.

$$W = \begin{vmatrix} 1-x^2 & x \\ -2x & 1 \end{vmatrix} = 1 - x^2 + 2x^2 = 1 + x^2 \neq 0$$

EVER!

(c) Use what you've learned in parts (a) and (b) to find the solution of the IVP $(x^2+1)y'' - 2xy' + 2y = 0$; $y(1) = 1$, $y'(1) = -1$.

$y(x) = c_1(1-x^2) + c_2 x$

$y(1) = 1 \Rightarrow c_2 = 1$

$y'(x) = -2x c_1 + c_2$

$y'(1) = -1 \Rightarrow -2c_1 + c_2 = -1$

$c_1 = 1$

$\boxed{y(x) = 1 - x^2 + x}$

(d) Is your solution in part (c) unique? Explain.

YES; IN STANDARD FORM, THE EQUATION IS

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = 0.$$

THEOREM FROM CLASS/TEXT

THE COEFFICIENT FUNCTIONS ARE CONTINUOUS EVERYWHERE,

THEREFORE THE IVP HAS A UNIQUE SOL'N FOR ANY CHOICE OF INITIAL CONDITIONS.

9. (12 points) An object is launched into the air so that at any time t, the object's velocity satisfies the equation $v' = -1.2v - 9.8$, where t is measured in seconds and v in meters per second. If the object is launched from the ground with an initial velocity of 50 m/s, find the equation of motion of the object. That is, find a function giving the object's position at time t.

$$v' + 1.2v = -9.8, \quad v(0) = 50$$

$$\mu(t) = e^{\int 1.2\,dt} = e^{1.2t}$$

$$e^{1.2t} v = \int -9.8 e^{1.2t}\,dt$$

$$e^{1.2t} v = -\frac{9.8}{1.2} e^{1.2t} + C$$

$$v(t) = -\frac{9.8}{1.2} + Ce^{-1.2t}$$

$$v(0) = 50 \Rightarrow -\frac{9.8}{1.2} + C = 50$$

$$C = \frac{349}{6}$$

$$v(t) = -\frac{49}{6} + \frac{349}{6} e^{-1.2t}$$

$$x(t) = C_1 - \frac{49}{6} t - \frac{1745}{36} e^{-1.2t}$$

$$x(0) = 0 \Rightarrow C_1 = \frac{1745}{36}$$

$$\boxed{x(t) = \frac{1745}{36} - \frac{49}{6} t - \frac{1745}{36} e^{-1.2t}}$$

10. (12 points) Solve the initial value problem $\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$, $y(1) = 1$. (Hint: The equation is homogeneous. Use the substitution $u = y/x$.)

$$u = \frac{y}{x}$$

$$y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = u + e^u$$

$$x\frac{du}{dx} = e^u$$

$$e^{-u}\,du = \frac{1}{x}\,dx$$

$$-e^{-u} = \ln|x| + C_1$$

$$e^{-u} = C_2 - \ln|x|$$

$$-u = \ln(C_2 - \ln|x|)$$

$$-\frac{y}{x} = \ln(C_2 - \ln|x|)$$

$$y(x) = -x\ln(C_2 - \ln|x|)$$

$$y(1) = 1 \Rightarrow -\ln C_2 = 1$$

$$\Rightarrow C_2 = e^{-1}$$

$$\boxed{y(x) = -x\ln(e^{-1} - \ln|x|)}$$

11. (6 points) Determine the recursive formula for the Taylor method of order 2 for the initial value problem.

$$y' = 4x^3y, \quad y(0) = 5$$

$f(x,y) = 4x^3y$ ✓ $\frac{dy}{dx} = 4x^3y$

$f'(x,y) = 12x^2y + 4x^3\frac{dy}{dx}$

$= 12x^2y + 16x^6y$

$$\boxed{\begin{aligned} y_{N+1} &= y_N + h\left(4x_N^3y_N\right) + \frac{h^2}{2}\left(12x_N^2y_N + 16x_N^6y_N\right) \\ x_{N+1} &= x_N + h \end{aligned}}$$

12. (10 points) Solve: $x''' - 2x'' + 3x' = 0.$

$r^3 - 2r^2 + 3r = 0$

$r(r^2 - 2r + 3) = 0$

$r = 0$ OR $\underbrace{r = \frac{2 \pm \sqrt{4-12}}{2}}$

$x_1 = e^{0t} = \text{CONST}$ $\quad \underbrace{r = 1 \pm \sqrt{2}\,i}$

$x_2 = e^t\cos\sqrt{2}\,t \quad x_3 = e^t\sin\sqrt{2}\,t$

$$\boxed{x(t) = c_1 + c_2e^t\cos\sqrt{2}\,t + c_3e^t\sin\sqrt{2}\,t}$$