Math 233 - Quiz 3

September 23, 2021

Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary. This quiz is due September 28.

1. (3 points) Find the vector-valued function $\vec{r}(t)$ such that

$$\vec{r}\,'(t) = \frac{5}{t^2+1}\hat{\imath} - \frac{3t}{(t^2+1)^2}\hat{\jmath} + e^{2t}\hat{k}, \qquad \vec{r}(0) = -\hat{\imath} + 3\hat{\jmath} + 2\hat{k}.$$

$$\vec{r}(t) = \int \frac{5}{t^2+1}\,dt\ \hat{\imath} - \int \frac{3t}{(t^2+1)^2}\,dt\ \hat{\jmath} + \int e^{2t}\,dt\ \hat{k}$$

$u = t^2+1$
$du = 2t\,dt$ $\quad \frac{3}{2}\int u^{-2}\,du$

$$\vec{r}(t) = (5\tan^{-1}t + c_1)\hat{\imath} + \left[\frac{3}{2}(t^2+1)^{-1} + c_2\right]\hat{\jmath} + \left(\frac{1}{2}e^{2t} + c_3\right)\hat{k}$$

$$\vec{r}(0) = -\hat{\imath} + 3\hat{\jmath} + 2\hat{k} = c_1\hat{\imath} + \left(\frac{3}{2} + c_2\right)\hat{\jmath} + \left(\frac{1}{2} + c_3\right)\hat{k}$$

$$\Rightarrow \quad c_1 = -1,\ c_2 = \frac{3}{2},\ c_3 = \frac{3}{2}$$

$$\vec{r}(t) = (5\tan^{-1}t - 1)\hat{\imath} + \left(\frac{3}{2(t^2+1)} + \frac{3}{2}\right)\hat{\jmath} + \left(\frac{1}{2}e^{2t} + \frac{3}{2}\right)\hat{k}$$

2. (2 points) Let $\vec{r}(t) = 3e^t\hat{\imath} + 2e^{-3t}\hat{\jmath} + 4e^{2t}\hat{k}$. Compute the unit tangent vector at $t = \ln 2$.

$$\vec{r}\,'(t) = 3e^t\hat{\imath} - 6e^{-3t}\hat{\jmath} + 8e^{2t}\hat{k}$$

$$\vec{r}\,'(\ln 2) = 3(2)\hat{\imath} - 6\left(\frac{1}{8}\right)\hat{\jmath} + 8(4)\hat{k}$$

$$= 6\hat{\imath} - \frac{3}{4}\hat{\jmath} + 32\hat{k}$$

$$\|\vec{r}\,'(\ln 2)\| = \sqrt{36 + \frac{9}{16} + 1024}$$

$$= \sqrt{\frac{16969}{16}} = \frac{\sqrt{16969}}{4}$$

$$\hat{T}(\ln 2) = \frac{24\hat{\imath} - 3\hat{\jmath} + 128\hat{k}}{\sqrt{16969}}$$

$$\approx 0.184\hat{\imath} - 0.023\hat{\jmath} + 0.983\hat{k}$$

Turn over.

3. (2 points) Reparameterize the position vector below in terms of the arc length parameter.

$$\vec{r}(t) = (1-3t)\hat{\imath} + (7+6t)\hat{\jmath} - 8t\hat{k}, \qquad 0 \le t \le 5$$

$\vec{r}\,'(t) = -3\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$, $\|\vec{r}\,'(t)\| = \sqrt{9+36+64} = \sqrt{109}$

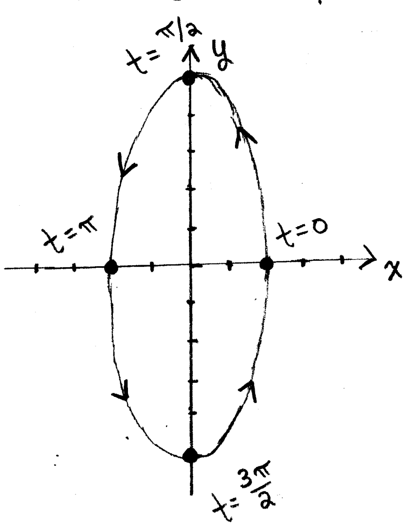
$S = \int_0^t \sqrt{109}\, d\tau = \sqrt{109}\, t$

$t = \dfrac{S}{\sqrt{109}}$ $\qquad \vec{R}(s) = \left(1 - \dfrac{3s}{\sqrt{109}}\right)\hat{\imath} + \left(7 + \dfrac{6s}{\sqrt{109}}\right)\hat{\jmath} - \dfrac{8s}{\sqrt{109}}\hat{k}$;

$0 \le S \le 5\sqrt{109}$

4. (3 points) The parametric equations below describe an ellipse in the plane. Draw a rough sketch of the ellipse (showing the orientation). Then use your intuition to determine a point where the curvature is greatest. Finally, compute the curvature at that point.

$$x = 2\cos t, \quad y = 5\sin t; \qquad 0 \le t < 2\pi$$



CURVATURE IS GREATEST AT $t = \frac{\pi}{2}$ AND $t = \frac{3\pi}{2}$.

THE CURVATURE IS EQUAL AT THOSE POINTS.

LET'S COMPUTE $K(\frac{\pi}{2})$.

$\vec{r}(t) = 2\cos t\,\hat{\imath} + 5\sin t\,\hat{\jmath}$

$\vec{r}\,'(t) = -2\sin t\,\hat{\imath} + 5\cos t\,\hat{\jmath}$ $\qquad \vec{r}\,'(\frac{\pi}{2}) = -2\hat{\imath}$

$\vec{r}\,''(t) = -2\cos t\,\hat{\imath} - 5\sin t\,\hat{\jmath}$ $\qquad \vec{r}\,''(\frac{\pi}{2}) = -5\hat{\jmath}$

$$\vec{r}\,'(\tfrac{\pi}{2}) \times \vec{r}\,''(\tfrac{\pi}{2}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & 0 \\ 0 & -5 & 0 \end{vmatrix} = 10\hat{k}$$

$$K(\tfrac{\pi}{2}) = \frac{\|\vec{r}\,'(\frac{\pi}{2}) \times \vec{r}\,''(\frac{\pi}{2})\|}{\|\vec{r}\,'(\frac{\pi}{2})\|^3} = \frac{10}{8} = \boxed{\frac{5}{4}}$$