Math 233 - Test 1
September 16, 2021

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Find a vector of magnitude 7 that has the direction from P $(2,5,8)$ to Q $(-1,3,4)$.

$\vec{PQ} = -3\hat{i} - 2\hat{j} - 4\hat{k}$

$\|\vec{PQ}\| = \sqrt{9+4+16} = \sqrt{29}$

$\dfrac{7\,\vec{PQ}}{\|\vec{PQ}\|} = \dfrac{-7}{\sqrt{29}}\left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$

2. (6 points) Let $\vec{u} = -5\hat{\imath} + 4\hat{\jmath} - \hat{k}$.

(a) Find a vector, different from $\vec{u}$, that is parallel to $\vec{u}$. Give a one-sentence explanation of how you know.

$-2\vec{u} = \boxed{10\hat{i} - 8\hat{j} + 2\hat{k}}$

TWO VECTORS ARE PARALLEL WHEN ONE IS A SCALAR MULTIPLE OF THE OTHER.

(b) Find a nonzero vector that is orthogonal to $\vec{u}$. Give a one-sentence explanation of how you know.

$\vec{v} = \boxed{4\hat{i} + 5\hat{j} + 0\hat{k}}$

$\vec{u} \cdot \vec{v} = (-5)(4) + (4)(5) + (-1)(0)$
$= 0$

TWO VECTORS ARE ORTHOGONAL WHEN THEIR DOT PRODUCT IS ZERO.

3. (2 points) Without actually computing the cross product, use the right-hand rule to determine $\hat{k} \times \hat{\jmath}$.

RIGHT HAND ...
FINGERS IN DIRECTION OF POS Z-AXIS.
CURL FINGERS TOWARD POS Y-AXIS.
THUMB POINTS IN DIRECTION OF NEG. X-AXIS.

$\hat{k} \times \hat{j} = -\hat{i}$

4. (6 points) Find a set of symmetric equations for the line passing through $(2, 1, 5)$ and parallel to the line with parametric equations $x = 3 + 2t$, $y = -6 - t$, $z = 7 - 5t$.

POINT: $(2,1,5)$

DIRECTION:

$\vec{v} = 2\hat{i} - \hat{j} - 5\hat{k}$.

LINE IS

$$\boxed{\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-5}{-5}}$$

5. (9 points) In the following problems, the force vectors $\vec{F_1}$ and $\vec{F_2}$ are 2D vectors in the xy-plane.

(a) The force $\vec{F_1}$ has magnitude 3 and makes a 225° angle with the positive x-axis. Find the component form of $\vec{F_1}$.

$$\vec{F_1} = 3\cos 225^\circ\,\hat{i} + 3\sin 225^\circ\,\hat{j}$$

$$= \boxed{-\frac{3\sqrt{2}}{2}\hat{i} - \frac{3\sqrt{2}}{2}\hat{j}}$$

(b) The force $\vec{F_2}$ has component form $\vec{F_2} = 5\sqrt{3}\hat{\imath} + 10\hat{\jmath}$. Compute the resultant vector $\vec{F} = \vec{F_1} + \vec{F_2}$.

$$\vec{F} = \left(5\sqrt{3} - \frac{3\sqrt{2}}{2}\right)\hat{i} + \left(10 - \frac{3\sqrt{2}}{2}\right)\hat{j}$$

$$\approx 6.54\,\hat{i} + 7.88\,\hat{j}$$

(c) What angle does $\vec{F}$ make with the positive x-axis?

$\vec{F}$ IS IN QUAD I AND $\tan\theta = \dfrac{10 - \frac{3\sqrt{2}}{2}}{5\sqrt{3} - \frac{3\sqrt{2}}{2}} \approx 1.205$

$$\Rightarrow \boxed{\theta \approx 50.31^\circ}$$

6. (4 points) The angle between $\vec{u}$ and $\vec{v}$ is obtuse. What can be said about $\vec{u} \cdot \vec{v}$? Briefly explain.

$\vec{u} \cdot \vec{v}$ IS NEGATIVE, BECAUSE

$\vec{u} \cdot \vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$, AND $\cos\theta < 0$ FOR OBTUSE $\angle$'s.

NONNEG. NON NEG.

7. (8 points) Find the angle between the planes described by the following equations.

$$2x - y + 2z = 7 \qquad\qquad -5x + 3z = 12$$

$\vec{n}_1 = 2\hat{i} - \hat{j} + 2\hat{k}$ $\qquad \|\vec{n}_1\| = \sqrt{4+1+4} = \sqrt{9} = 3$

$\vec{n}_2 = -5\hat{i} + 3\hat{k}$ $\qquad \|\vec{n}_2\| = \sqrt{25+9} = \sqrt{34}$

$\vec{n}_1 \cdot \vec{n}_2 = -10 + 6$

$= -4$

$\cos\theta = \dfrac{|-4|}{3\sqrt{34}} \Rightarrow \boxed{\theta \approx 76.78°}$

8. (6 points) Let $\vec{r}(t) = \dfrac{\sin t}{t}\hat{\imath} + \ln(t+1)\hat{\jmath} + e^{2t}\hat{k}$.

(a) Determine the domain of $\vec{r}$.

$\underbrace{(t \neq 0)}_{\hat{i}} \cap \underbrace{(t > -1)}_{\hat{j}} \cap \underbrace{\mathbb{R}}_{\hat{k}}$

$\boxed{\text{DOMAIN} = \{t : t > -1 \text{ AND } t \neq 0\}}$

(b) Compute $\lim_{t \to 0} \vec{r}(t)$.

$= 1\hat{i} + \ln 1\,\hat{j} + e^0\hat{k}$

$= \boxed{\hat{i} + \hat{k}}$

9. (6 points) Find the projection of $\vec{w} = \hat{\imath} + 4\hat{\jmath} - 3\hat{k}$ onto $\vec{u} = 7\hat{\imath} + 4\hat{k}$.

$\text{proj}_{\vec{u}}\,\vec{w} = \dfrac{\vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\vec{u} = \dfrac{7+0-12}{49+16}\vec{u} = \dfrac{-5}{65}\vec{u}$

$= \boxed{-\dfrac{1}{13}\vec{u} = -\dfrac{7}{13}\hat{i} - \dfrac{4}{13}\hat{k}}$

10. (8 points) Find a set of parametric equations for a line in the plane $5x - 9y - 8z = 5$.

TWO POINTS IN THE PLANE ARE $(1,0,0)$ AND $(0,-5,5)$.

P Q

$\overleftrightarrow{PQ}$ IS A LINE IN THE PLANE.

$\vec{PQ} = -\hat{i} - 5\hat{j} + 5\hat{k}$

POINT $(1,0,0)$

$$\boxed{\begin{aligned} x &= 1-t \\ y &= -5t \\ z &= 5t \end{aligned}}$$

11. (7 points) Find the vector-valued function $\vec{r}(t)$ such that

$$\vec{r}\,'(t) = te^{-t^2}\hat{\imath} - e^{-t}\hat{\jmath} + \hat{k}; \quad \vec{r}(0) = \frac{1}{2}\hat{\imath} - \hat{\jmath} + 2\hat{k}.$$

INTEGRATE COMPONENT-BY-COMPONENT...

$\hat{i}$: $\int te^{-t^2}\,dt = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^{-t^2} + C_1$

$u = -t^2$

$du = -2t\,dt$

$\hat{j}$: $\int -e^{-t}\,dt = e^{-t} + C_2$

$\hat{k}$: $\int 1\,dt = t + C_3$

$\vec{r}(t) = \left(-\frac{1}{2}e^{-t^2} + C_1\right)\hat{i} + \left(e^{-t} + C_2\right)\hat{j} + \left(t + C_3\right)\hat{k}$

$\vec{r}(0) = \left(-\frac{1}{2} + C_1\right)\hat{i} + \left(1 + C_2\right)\hat{j} + C_3\hat{k}$

$= \frac{1}{2}\hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow C_1 = 1,\ C_2 = -2,\ C_3 = 2$

$$\boxed{\vec{r}(t) = \left(-\frac{1}{2}e^{-t^2} + 1\right)\hat{i} + \left(e^{-t} - 2\right)\hat{j} + \left(t + 2\right)\hat{k}}$$

12. (6 points) Sketch the graph of the vector-valued function $\vec{r}(t) = t^3\hat{\imath} + t\hat{\jmath}$. Draw arrows on your graph to indicate the curve's orientation.

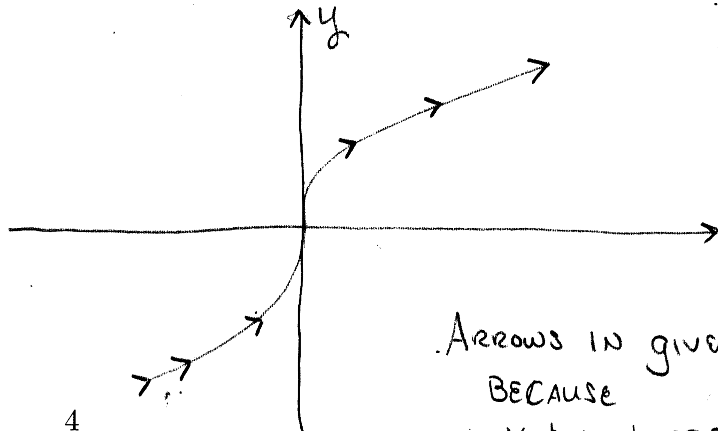
$\vec{r}(t) = t^3\hat{i} + t\hat{j}$

$\Downarrow$

$x = t^3$

$y = t$.

OR $y = \sqrt[3]{x}$



ARROWS IN GIVEN DIRECTION BECAUSE x & y INCREASE WITH t.

13. (8 points) Let $\vec{r}(t) = \cos 2t\,\hat{\imath} - \sin 2t\hat{\jmath} + 4t\hat{k}$.

(a) Let $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$. Compute $\hat{T}(t)$.

$$\vec{r}\,'(t) = -2\sin 2t\,\hat{\imath} - 2\cos 2t\,\hat{\jmath} + 4\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 16}$$

$$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\boxed{\hat{T}(t) = \frac{1}{\sqrt{5}}\left(-\sin 2t\,\hat{\imath} - \cos 2t\,\hat{\jmath} + 2\hat{k}\right)}$$

(b) Compute $\hat{T}(t) \cdot \hat{T}\,'(t)$.

$$\hat{T}\,'(t) = -\frac{2}{\sqrt{5}}\cos 2t\,\hat{\imath} + \frac{2}{\sqrt{5}}\sin 2t$$

$$\hat{T}(t) \cdot \hat{T}\,'(t) = \frac{2}{5}\cos 2t \sin 2t - \frac{2}{5}\cos 2t \sin 2t + 0$$

$$= \boxed{0}$$

14. (10 points) Let $\vec{u} = -2\hat{\imath} + 9\hat{\jmath} + \hat{k}$ and $\vec{v} = \hat{\imath} - \hat{\jmath} + 4\hat{k}$.

(a) Find a vector orthogonal to both $\vec{u}$ and $\vec{v}$.

THE EASY CHOICE IS $\vec{0}$, BUT THIS WON'T HELP WITH PART (b).

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 9 & 1 \\ 1 & -1 & 4 \end{vmatrix} = \hat{\imath}(36+1) - \hat{\jmath}(-8-1) + \hat{k}(2-9)$$

$$= \boxed{37\hat{\imath} + 9\hat{\jmath} - 7\hat{k}}$$

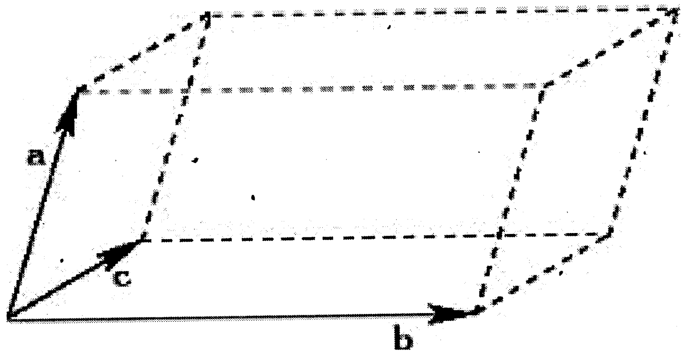
(b) Find an equation of the plane passing through $(5, 0, 3)$ with normal vector is $\vec{u} \times \vec{v}$.

$$37(x-5) + 9(y-0) - 7(z-3) = 0$$

OR

$$\boxed{37x + 9y - 7z = 164}$$

15. (8 points) A crystal structure has the form of a parallelepiped determined by the vectors $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = 3\hat{\jmath} + 5\hat{k}$, and $\vec{c} = -4\hat{\imath} + 2\hat{\jmath} + \hat{k}$, where distances are measured in micrometers. Find the volume of the parallelepiped.



$$\text{Volume} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ -4 & 2 & 1 \end{vmatrix} = 1(3-10) - 2(0+20) + 1(0+12)$$

$$= -7 - 40 + 12 = -35$$

$$\boxed{\text{Volume} = 35\,\mu m^3}$$