Math 233 - Test 2

October 14, 2021

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (18 points) In a field goal attempt on a flat field, a football is kicked off the ground at an angle of 30° to the horizontal with an initial speed of 65 ft/sec. (Ignore air resistance in this problem and use $g = 32\,\text{ft/s}^2$.)

   (a) What horizontal distance does the football travel while it is in the air?

$$\vec{r}(t) = 65\cos 30^\circ\, t\,\hat{\imath} + \left(-16t^2 + 65\sin 30^\circ t\right)\hat{\jmath}$$
$$= \frac{65\sqrt{3}}{2}t\,\hat{\imath} + \left(-16t^2 + \frac{65}{2}t\right)\hat{\jmath}$$

$$-16t^2 + \frac{65}{2}t = 0 \Rightarrow t = \frac{\frac{65}{2}}{16} = 2.03125\text{ s}$$

$$\left(\frac{65\sqrt{3}}{2}\right)(2.03125) \approx \boxed{114.34\text{ FT}}$$

   (b) Set up the definite integral that gives the length of the path of the football. Use your calculator to estimate the value of the integral.

$$\vec{r}\,'(t) = \frac{65\sqrt{3}}{2}\hat{\imath} + \left(-32t + \frac{65}{2}\right)\hat{\jmath}$$

$$\text{ARC LENGTH} = \int_0^{2.03125} \sqrt{\left(\frac{65\sqrt{3}}{2}\right)^2 + \left(-32t + \frac{65}{2}\right)^2}\,dt \approx 120.41\text{ FT}$$

   (c) To score a field goal, the ball must clear the cross bar of the goal post, which is 10 ft above the ground. What is the furthest from the goal post the kick can originate and score a field goal?

COMPUTE $\frac{65\sqrt{3}}{2}t$ WHEN $\underbrace{-16t^2 + \frac{65}{2}t = 10}$

QUAD FORMULA...

$$t = \frac{-32.5 - \sqrt{(32.5)^2 - 4(-16)(-10)}}{-32}$$

$$\approx 1.6532\text{ s}$$

$$\left(\frac{65\sqrt{3}}{2}\right)(1.6532) \approx \boxed{93.06\text{ FT}}$$

2. (12 points) An object is moving in such a way that its position at time t is given by

$$\vec{r}(t) = \sin 3t\,\hat{\imath} - \cos 3t\,\hat{\jmath} + 2t^2\hat{k}.$$

(a) Determine the function that gives the speed of the object at time t.

$\vec{r}'(t) = 3\cos 3t\hat{\imath} + 3\sin 3t\hat{\jmath} + 4t\hat{k}$

$\|\vec{r}'(t)\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 16t^2} = \sqrt{9+16t^2}$

$\boxed{\|\vec{r}'(t)\| = \sqrt{9+16t^2}}$

(b) Find the speed at time $t = 5$.

$\|\vec{r}'(5)\| = \sqrt{9+400} = \boxed{\sqrt{409} \approx 20.22}$

(c) Find $\hat{T}(t)$, the unit tangent vector for $\vec{r}(t)$.

$$\boxed{\hat{T}(t) = \frac{3\cos 3t\,\hat{\imath} + 3\sin 3t\,\hat{\jmath} + 4t\,\hat{k}}{\sqrt{9+16t^2}}}$$

(d) What would you find if you computed $\hat{T}(t) \cdot \hat{T}'(t)$? Explain.

$\hat{T}(t) \cdot \hat{T}'(t) = 0$

A VVF OF CONSTANT MAGNITUDE ($\|\hat{T}(t)\| = 1$) IS ORTHOGONAL TO ITS DERIVATIVE.

3. (8 points) Let $\vec{r}(t) = -\cos 7t\,\hat{\imath} - \sin 7t\,\hat{\jmath} + t\hat{k}$. Compute $\hat{N}(t)$.

$\vec{r}'(t) = 7\sin 7t\,\hat{\imath} - 7\cos 7t\,\hat{\jmath} + \hat{k}$

$\|\vec{r}'(t)\| = \sqrt{49+1} = \sqrt{50}$

$\hat{T}(t) = \frac{1}{\sqrt{50}}\left(7\sin 7t\,\hat{\imath} - 7\cos 7t\,\hat{\jmath} + \hat{k}\right)$

$\hat{T}'(t) = \frac{1}{\sqrt{50}}\left(49\cos 7t\hat{\imath} + 49\sin 7t\,\hat{\jmath}\right)$

$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \cos 7t\,\hat{\imath} + \sin 7t\,\hat{\jmath}$

4. (8 points) Consider the vector-valued function $\vec{r}(t) = 2t\hat{\imath} - t^2\hat{\jmath}$.

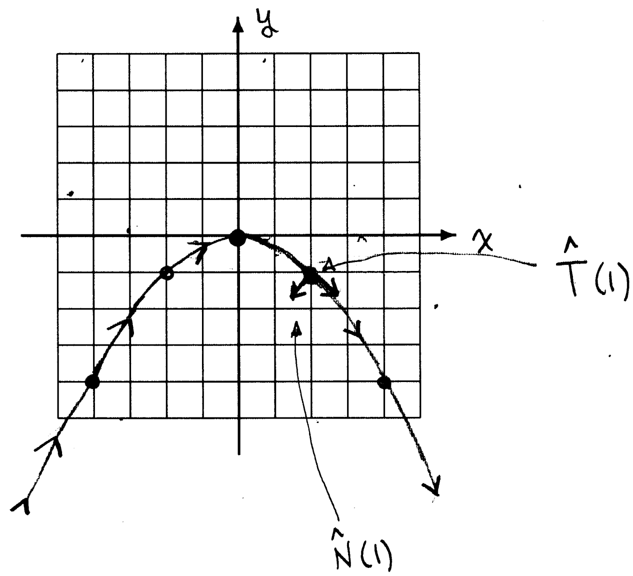
(a) Sketch the graph of $\vec{r}(t)$. Show or describe the orientation of the curve.

$x = 2t$

$y = -t^2$

$\Downarrow$

$y = -\left(\frac{x}{2}\right)^2$



(b) Without actually computing $\hat{T}(t)$, sketch the vector $\hat{T}(1)$ on your graph. Label it.

$\hat{T}(1)$ IS TANGENT TO THE GRAPH, IN THE DIRECTION OF MOTION, AND WITH MAGNITUDE 1

(c) Without actually computing $\hat{N}(t)$, sketch the vector $\hat{N}(1)$ on your graph. Label it.

$\hat{N}(1)$ IS PERPENDICULAR TO $\hat{T}(1)$, POINTS TOWARD THE CONCAVE SIDE OF CURVE, AND HAS MAGNITUDE 1.

(d) How would your sketch of $\hat{T}(1)$ be different if the first component of $\vec{r}(t)$ was $-2t$ rather than $2t$? Explain.

THE GRAPH WOULD BE THE SAME, BUT THE ORIENTATION WOULD BE REVERSED. WITH $x = -2t$, THE X-COORDINATES DECREASE WITH INCREASING t.

5. (3 points) Determine the domain of the function $F(x, y, z) = \ln(1 - x^2 - y^2 - z^2)$.

$$1 - x^2 - y^2 - z^2 > 0 \Rightarrow x^2 + y^2 + z^2 < 1$$

DOMAIN IS THE SET OF ALL PTS IN $\mathbb{R}^3$ INSIDE THE SPHERE $x^2 + y^2 + z^2 = 1$.

6. (6 points) For a smooth curve described by $\vec{r}(t)$, the following facts are known:

- $\hat{T}(t_0) = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{k}$
- $\hat{T}'(t_0) = -4\hat{\imath} - 3\hat{k}$
- $\|\vec{r}'(t_0)\| = 10$

(a) Find $\vec{r}'(t_0)$.

$$\vec{r}'(t_0) = \|\vec{r}'(t_0)\|\, \hat{T}(t_0) = 10 \cdot \left(\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{k}\right) = \boxed{6\hat{\imath} - 8\hat{k}}$$

(b) Find $\hat{N}(t_0)$.

$$\hat{N}(t_0) = \frac{\hat{T}'(t_0)}{\|\hat{T}'(t_0)\|} = \frac{-4\hat{\imath} - 3\hat{k}}{\sqrt{16+9}} = -\frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{k}$$

7. (15 points) Determine the limit or show that it does not exist.

(a) $\displaystyle \lim_{(x,y)\to(2,0)} \frac{(x-y)^2 + 2(x-y) - x^2 - 2x}{y}$ % MORE WORK.

$$= \lim_{(x,y)\to(2,0)} \frac{x^2 - 2xy + y^2 + 2x - 2y - x^2 - 2x}{y} = \lim_{(x,y)\to(2,0)} \frac{-2xy + y^2 - 2y}{y}$$

$$= \lim_{(x,y)\to(2,0)} (-2x + y - 2) = \boxed{-6}$$

(b) $\displaystyle \lim_{(x,y)\to(1,2)} \frac{xy^3 - 8x^2}{xy^2 - 4}$ % MORE WORK.

ALONG $x = 1$:

$$\lim_{y\to 2} \frac{y^3 - 8}{y^2 - 4} = \lim_{y\to 2} \frac{3y^2}{2y} = 3$$

ALONG $y = 2$:

$$\lim_{x\to 1} \frac{8x - 8x^2}{4x - 4} = \lim_{x\to 1} \frac{8x(1-x)}{4(x-1)} = -2$$

$\boxed{\text{LIMIT DNE}}$

(c) $\displaystyle \lim_{(x,y)\to(1,1)} \frac{x-y}{\sqrt{x} - \sqrt{y}}$ % MORE WORK.

$$\lim_{(x,y)\to(1,1)} \frac{x-y}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{x-y}} = \boxed{2}$$

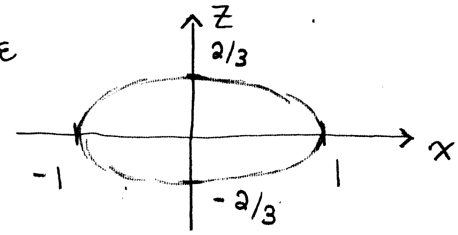
8. (6 points) Consider the surface described by the equation $4x^2 - y^2 + 9z^2 = 4$.

(a) Fix a value for one of the variables and draw a good sketch of the corresponding level curve.

$$y = 0 \Rightarrow 4x^2 + 9z^2 = 4$$

$$x^2 + \frac{z^2}{\left(\frac{2}{3}\right)^2} = 1.$$

ELLIPSE



(b) Fix a value for one of the other variables and briefly describe the corresponding level curve.

$$z = 0 \Rightarrow 4x^2 - y^2 = 4$$ HYPERBOLA IN THE XY-PLANE

$$x = 0 \Rightarrow 9z^2 - y^2 = 4$$ HYPERBOLA IN THE YZ-PLANE

(c) Identify the surface.

$$4x^2 + 9z^2 = 4 + y^2$$

FOR ANY FIXED y, THESE ARE ELLIPSES.

HYPERBOLOID OF ONE SHEET

9. (6 points) Consider the function $h(x, y) = \sqrt{1 + x - y^2}$.

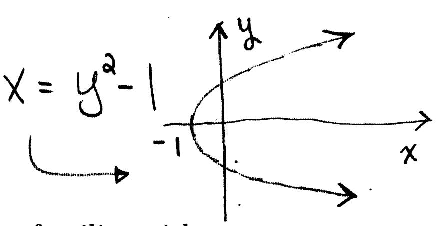
(a) What is the domain of h?

$$1 + x - y^2 \geq 0$$

$$\boxed{\{(x,y) \in \mathbb{R}^2 : x \geq y^2 - 1\}}$$

(b) Sketch the level curve $h(x, y) = 0$.

$$h(x,y) = 0 \Rightarrow 1 + x - y^2 = 0 \Rightarrow x = y^2 - 1$$



(c) The graph of h is one-half of one of the quadric surfaces that we are familiar with. Describe the graph of h.

$$z^2 = 1 + x - y^2$$

OR

$$x = y^2 + z^2 - 1$$

THE SURFACE IS THE $z \geq 0$ HALF OF A PARABOLOID, OPENING UP THE X AXIS, WITH VERTEX AT $(-1, 0, 0)$.

10. (8 points) Starting from $t = 0$, compute the arc-length parameter for the curve described by
$$\vec{r}(t) = \frac{1}{2}t^2\,\hat{\imath} + \frac{1}{3}(2t+1)^{3/2}\,\hat{\jmath} + \hat{k}.$$

$$\vec{r}\,'(t) = t\,\hat{\imath} + \frac{1}{3}\cdot\frac{3}{2}(2t+1)^{1/2}(2)\,\hat{\jmath}$$
$$= t\,\hat{\imath} + \sqrt{2t-1}\,\hat{\jmath}$$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2+2t+1}$$
$$= t+1$$

$$s(t) = \int_0^t (\tau+1)\,d\tau$$
$$= \frac{1}{2}\tau^2 + \tau \Big|_0^t$$

$$\boxed{s(t) = \frac{1}{2}t^2 + t}$$

11. (2 points) Describe the graph of a vector-valued function for which the curvature is 0 for all values of t.

THE GRAPH IS A LINE.

12. (2 points) Describe the graph of a vector-valued function for which the curvature is 1 for all values of t.

THE GRAPH IS A CIRCLE OF RADIUS 1.

13. (6 points) Let a be a positive real number and let $\vec{r}(t) = t\,\hat{\imath} + at^2\,\hat{\jmath}$. The graph of $\vec{r}$ is a parabola. Find the curvature function.

$$\vec{r}\,'(t) = \hat{\imath} + 2at\,\hat{\jmath}$$
$$\vec{r}\,''(t) = 2a\,\hat{\jmath}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2a & 0 \\ 0 & 2a & 0 \end{vmatrix}$$
$$= 2a\,\hat{k}$$

$$\boxed{K = \frac{2a}{(1+4a^2t^2)^{3/2}}}$$

$$\|\vec{r}\,'(t)\| = \sqrt{1+4a^2t^2}$$