# Math 233 - Test 3a

November 11, 2021

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Discuss the continuity of each function. Explain your reasoning.

   (a) $f(x, y) = \ln(x + y)$

   $\ln x$ IS CONTINUOUS FOR ALL $X > 0$

   $X+y$ IS A POLYNOMIAL--- CONTINUOUS EVERYWHERE

   $f$ IS CONT FOR $X+y > 0$.

   (b) $g(x, y, z) = e^{5xyz}$

   $e^X$ IS CONTINUOUS EVERYWHERE

   $5xyz$ IS A POLYNOMIAL--- CONT. EVERYWHERE.

   $g$ IS CONT. EVERYWHERE ($\mathbb{R}^3$)

2. (8 points) Let $z = \sin(x^2y - 2x + 4)$.

   (a) Compute $\partial z/\partial x$.

   $$\frac{\partial z}{\partial x} = \cos(x^2y - 2x + 4)(2xy - 2)$$

   (b) Compute $\partial z/\partial y$.

   $$\frac{\partial z}{\partial y} = \cos(x^2y - 2x + 4)(x^2)$$

   (c) If you were on the graph of the equation at the point $(2, 0, 0)$ and you looked in the direction of the positive $x$-axis, would you be looking uphill or downhill? Explain.

   $$\left.\frac{\partial z}{\partial x}\right|_{(2,0,0)} = \cos(0) \cdot (-2) = -2 \Rightarrow \text{DOWNHILL---}$$

   SLOPE IS NEG.

   (d) If you were on the graph of the equation at the point $(2, 0, 0)$ and you looked in the direction of the positive $y$-axis, would you be looking uphill or downhill? Explain.

   $$\left.\frac{\partial z}{\partial y}\right|_{(2,0,0)} = \cos(0) \cdot (2^2) = 4 \Rightarrow \text{UPHILL---}$$

   SLOPE IS POS.

3. (9 points) Suppose $z = 2xe^{5y} - 3ye^{-x}$.

(a) Which first partial derivative should be computed first in order to obtain $\frac{\partial^2 z}{\partial x \partial y}$?

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

THE y-PARTIAL IS COMPUTED FIRST.

(b) Do you expect to have $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$? Explain.

YES, $f(x,y) = 2xe^{5y} - 3ye^{-x}$ AND ALL PARTIAL DERIVATIVES OF ALL ORDERS WILL BE CONTINUOUS. BY OUR THEOREM, THE CONTINUITY OF THE $2^{\text{ND}}$ ORDER MIXED PARTIALS GUARANTEES THEIR EQUALITY.

(c) Compute $\frac{\partial^2 z}{\partial x \partial y}$.

$$\frac{\partial z}{\partial y} = 10xe^{5y} - 3e^{-x}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = 10e^{5y} + 3e^{-x}}$$

4. (8 points) Suppose $w = f(x, y)$, where $x = u - v$ and $y = v - u$. Use the chain rule to show that $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$.

ADD

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y}(-1)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial w}{\partial x}(-1) + \frac{\partial w}{\partial y}(1)$$

ADD THESE TO GET ZERO.

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

5. (6 points) Suppose $z$ is implicitly defined as a function of $x$ and $y$ by the equation

$$\underbrace{x \ln y + y^2 z + z^2}_{F(x,y,z)} = 8.$$

Find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(\ln y)}{y^2+2z}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\left(\frac{x}{y}+2yz\right)}{y^2+2z}}$$

6. (6 points) The temperature at the point $(x, y)$ on a metal plate is given by

$$T(x, y) = 4 + \sin(xy) + x + xy, \quad -2 \le x \le 2,\ -2 \le y \le 2.$$

(a) Find the direction of greatest increase in temperature from the point $(0, 1)$.

DIRECTION OF GRADIENT VECTOR AT $(0,1)$ ...

$\vec{\nabla} T(x,y) = (y\cos(xy)+1+y)\hat{i} + (x\cos(xy)+x)\hat{j}$

$\vec{\nabla} T(0,1) = 3\hat{i}$

DIRECTION OF $\vec{\nabla}T(0,1)$ IS THAT OF $\hat{i}$.

(b) At which point is there no increase or decrease in temperature regardless of which direction we look?

$\vec{\nabla}T(x,y) = \vec{0} \Rightarrow y\cos(xy)+1+y=0$ AND $x\cos(xy)+x = 0$

$x = 0$ OR $\cos(xy) = -1$

$\cos(xy) = -1$: $-y+1+y=0$ Nope!

OR $x = 0$: $2y+1=0$, $y = -\frac{1}{2}$

$\boxed{(0, -\tfrac{1}{2})}$

7. (5 points) Find an equation of the plane tangent to the surface $xy^2 + 3x - z^2 = 8$ at the point $(1, -3, 2)$.

$F(x,y,z) = xy^2 + 3x - z^2$

OUR SURFACE IS THE LEVEL SURFACE

$F(x,y,z) = 8$.

$\vec{\nabla} F(x,y,z) = (y^2+3)\hat{i} + (2xy)\hat{j} + (-2z)\hat{k}$

$\vec{n} = \vec{\nabla} F(1,-3,2) = 12\hat{i} - 6\hat{j} - 4\hat{k}$

POINT $(1,-3,2)$

TANGENT PLANE:

$12(x-1) - 6(y+3) - 4(z-2) = 0$

OR

$12x - 6y - 4z = 22$

8. (6 points) Find and classify the critical points of $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$.

$$f_x(x,y) = 2x - 2y - 2 = 0$$

$$f_y(x,y) = -2x + 4y + 2 = 0$$

$$2y = 0 \Rightarrow y = 0$$

$$2x - 2 = 0 \Rightarrow x = 1$$

$(1,0)$ IS THE ONLY CRIT PT.

$$d(x,y) = \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 8 - 4 = 4$$

$$d(1,0) = 4 > 0$$

AND

$$f_{xx}(1,0) = 2 > 0$$

$$\Downarrow$$

$f(1,0) = 0$ IS A REL MIN.

9. (6 points) Find the directional derivative of $f(x, y) = e^y \sin x$ at the point $(0, 0)$ in the direction toward the point $(2, 1)$.

$$\vec{\nabla} f(x,y) = e^y \cos x \,\hat{i} + e^y \sin x \,\hat{j}$$

$$\vec{\nabla} f(0,0) = \hat{i}$$

$$\vec{PQ} = 2\hat{i} + \hat{j}$$

$$\|\vec{PQ}\| = \sqrt{5}$$

$$\vec{\nabla} f(0,0) \cdot \frac{\vec{PQ}}{\|\vec{PQ}\|} = \boxed{\frac{2}{\sqrt{5}}}$$

Follow-up question: What does your directional derivative actually measure?

IT MEASURES THE SLOPE OF THE SURFACE $Z = e^y \sin X$ AT $(0,0)$ LOOKING IN THE DIRECTION OF $2\hat{i} + \hat{j}$.