Math 233 - Test 3b
November 11, 2021

Name: key

Score ______

Show all work to receive full credit. Supply explanations where necessary. You must work individually on this test. This test is due November 16.

1. (8 points) Find a set of parametric equations for the line normal to the surface $z = e^{4x^2+2xy-6y}$ at the point $(1, 1, 1)$.

$F(x,y,z) = e^{4x^2+2xy-6y} - z$

$\vec{\nabla} F(x,y,z) = (8x+2y)e^{4x^2+2xy-6y}\,\hat{i} + (2x-6)e^{4x^2+2xy-6y}\,\hat{j} - \hat{k}$

$\vec{\nabla} F(1,1,1) = 10\hat{i} - 4\hat{j} - \hat{k}$

$\vec{n} = 10\hat{i} - 4\hat{j} - \hat{k}$

POINT $(1,1,1)$ $\Rightarrow$

$$\boxed{\begin{aligned} x &= 1+10t \\ y &= 1-4t \\ z &= 1-t \end{aligned}}$$

2. (6 points) Find the linearization of $h(x, y) = \tan^{-1}(y/x)$ at the point $(1, 1)$. Then use the linearization to approximate $h(0.98, 1.03)$.

$h_x(x,y) = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\dfrac{y}{x^2}\right)$ $\quad h_x(1,1) = -\dfrac{1}{2}$

$h_y(x,y) = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\left(\dfrac{1}{x}\right)$ $\quad h_y(1,1) = \dfrac{1}{2}$

$h(1,1) = \text{TAN}^{-1}(1) = \dfrac{\pi}{4}$

$L(x,y) = \dfrac{\pi}{4} - \dfrac{1}{2}(x-1) + \dfrac{1}{2}(x-1)$

$h(0.98, 1.03) \approx L(0.98, 1.03) = \dfrac{\pi}{4} - \dfrac{1}{2}(-0.02) + \dfrac{1}{2}(0.03)$

$= \dfrac{\pi}{4} + 0.025 \approx 0.8103982$

$f_x(x,y) = 2x + 3y \quad , \quad f_y(x,y) = 3x - 4y - 1$

3. (8 points) Use the definition of differentiability to show that $f(x,y) = x^2 + 3xy - 2y^2 - y$ is differentiable everywhere on $\mathbb{R}^2$.

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= (x+\Delta x)^2 + 3(x+\Delta x)(y+\Delta y) - 2(y+\Delta y)^2 - (y+\Delta y) - x^2 - 3xy + 2y^2 + y$$

$$= 2x\Delta x + \Delta x^2 + 3x\Delta y + 3y\Delta x + 3\Delta x\,\Delta y - 4y\Delta y - 2\Delta y^2 - \Delta y$$

$$= (2x+3y)\Delta x + (3x-4y-1)\Delta y + (\Delta x + 3\Delta y)\Delta x + (-2\Delta y)\Delta y$$

$$= f_x\,\Delta x + f_y\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y.$$

WHERE $\epsilon_1 = \Delta x + 3\Delta y$

AND $\epsilon_2 = -2\Delta y$.

$\epsilon_1 \to 0$ AND $\epsilon_2 \to 0$ AS $(\Delta x, \Delta y) \to (0,0)$.

EVERYTHING ABOVE HOLDS FOR ANY (x,y) IN $\mathbb{R}^2$.

$\Downarrow$

f IS DIFFERENTIABLE EVERYWHERE.

4. (10 points) Find and classify the critical points of $g(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$.

$g_x(x,y) = -6xy - 6x = 0$

$g_y(x,y) = 3y^2 - 3x^2 - 6y = 0$

$\Rightarrow$ $-6x(y+1) = 0$

$x = 0$ or $y = -1$

$3y^2 - 6y = 0$ 　 $-3x^2 + 9 = 0$

$3y(y-2) = 0$ 　 $x = \pm\sqrt{3}$

$y = 0,\ y = 2$

Crit pts: $(0,0)$, $(0,2)$, $(\sqrt{3},-1)$, $(-\sqrt{3},-1)$

$$d(x,y) = \begin{vmatrix} -6y-6 & -6x \\ -6x & 6y-6 \end{vmatrix} = -36y^2 + 36 - 36x^2$$

$d(0,0) = 36$ AND $g_{xx}(0,0) = -6$ $\Rightarrow$ $g(0,0) = 1$ IS A REL. MAX

$d(0,2) = -108$ $\Rightarrow$ $(0,2,-3)$ IS A SADDLE PT.

$d(\sqrt{3},-1) = -108$ $\Rightarrow$ $(\sqrt{3},-1,-3)$ IS A SADDLE PT.

$d(-\sqrt{3},-1) = -108$ $\Rightarrow$ $(-\sqrt{3},-1,-3)$ IS A SADDLE PT.

5. (8 points) The *body mass index* (BMI) for an adult human is given by $B = 703w/h^2$, where w is weight in pounds and h is height in inches. Suppose you weigh 185 lbs and your height is 68 in. Compute your BMI. Then assume your weight and height measurements have errors $\Delta w = 1.75$ lbs and $\Delta h = 0.5$ in. Use differentials to estimate the error in your BMI.

$$B = \frac{(703)(185)}{(68)^2} \approx 28.126$$

$$\Delta B \approx \frac{\partial B}{\partial w}\Delta w + \frac{\partial B}{\partial h}\Delta h$$

$$= \frac{703}{h^2}\Delta w + \frac{-2(703w)}{h^3}\Delta h$$

$w = 185$ & $h = 68$

$$\Rightarrow \Delta B \approx \frac{703}{(68)^2}(1.75) - \frac{2(703)(185)}{(68)^3}(0.5)$$

$$\approx -0.1475613$$

$$\boxed{B \approx 28.126, \quad \Delta B \approx -0.148}$$