Math 233 - Final Exam A

December 9, 2021

Name key

Score ________

Show all work to receive full credit. Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation. Place your final answer in the box provided. This test is due December 14.

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1. Use the cross product to find the area of $\triangle ABC$, where

$$A(1,2,1), \quad B(3,-1,0), \quad C(2,1,-1).$$

$\vec{AB} = 2\hat{i} - 3\hat{j} - \hat{k}$

$\vec{AC} = \hat{i} - \hat{j} - 2\hat{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$= 5\hat{i} + 3\hat{j} + \hat{k}$

$\text{AREA} = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$

$= \frac{1}{2}\sqrt{25+9+1} = \frac{\sqrt{35}}{2}$

$$\boxed{\frac{\sqrt{35}}{2} \text{ UNITS}^2}$$

2. Let P_1 be the plane through the points A, B, and C of the problem above. Let P_2 be the plane described by the equation $4x + 3y - 2z = 7$. Find the angle between the planes P_1 and P_2.

$\vec{n}_1 = 5\hat{i} + 3\hat{j} + \hat{k}$

$\vec{n}_2 = 4\hat{i} + 3\hat{j} - 2\hat{k}$

$\|\vec{n}_2\| = \sqrt{16+9+4} = \sqrt{29}$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\,\|\vec{n}_2\|} = \frac{27}{\sqrt{35}\sqrt{29}} \approx 0.8475 \Rightarrow \theta \approx 0.56 \text{ RADIANS}$$

$\approx 32.06^\circ$

$$\boxed{0.56 \text{ RAD} \quad \text{OR} \quad 32.06^\circ}$$

3. In Moscow in 1987, Natalya Lisouskaya set a women's shot put world record by putting a shot 73.833 ft. She launched the shot at a 40° angle to the horizontal from 6.5 ft above the ground. What was the shot's initial speed? (To receive full credit, you must write and use the vector-valued function $\vec{r}(t)$ that gives the position of the shot at time t. Also ignore air resistance and use $g \approx 32\,\mathrm{ft\,s^{-2}}$.)

$$\vec{r}(t) = V_0 \cos 40^\circ t \,\hat{\imath} + \left(-16t^2 + V_0 \sin 40^\circ t + 6.5\right)\hat{\jmath}$$

$$V_0 \cos 40^\circ t = 73.833 \Rightarrow V_0 = \frac{73.833}{\cos 40^\circ t}.$$

$$-16t^2 + V_0 \sin 40^\circ t + 6.5 = 0$$

$$-16t^2 + 73.833 \tan 40^\circ + 6.5 = 0$$

$$t^2 \approx 4.2783\ldots \Rightarrow t \approx 2.0684 \text{ s}$$

$V_0 \approx 46.597$ FT/S

$$\boxed{V_0 \approx 46.597 \text{ FT/S}}$$

4. Set up the definite integral that gives the length of the space curve

$$\vec{r}(t) = 6t^3\hat{\imath} - 2t^2\hat{\jmath} + 5t\hat{k}$$

from the point $(0, 0, 0)$ to the point $(6, -2, 5)$. Then use technology to approximate the value of your integral.

$$\vec{r}\,'(t) = 18t^2\hat{\imath} - 4t\hat{\jmath} + 5\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{324t^4 + 16t^2 + 25}$$

$t = 0 \rightarrow (0,0,0)$

$t = 1 \rightarrow (6,-2,5)$

$$\boxed{\int_0^1 \sqrt{324t^4 + 16t^2 + 25}\,dt \approx 8.813}$$

5. Find the limit or show that it does not exist: $\lim_{(x,y)\to(1,1)} \frac{xy - y - 2x + 2}{3x - 3}$ 0/0 MORE WORK!

$$\lim_{(x,y)\to(1,1)} \frac{y(x-1) - 2(x-1)}{3(x-1)}$$

$$= \lim_{(x,y)\to(1,1)} \frac{y-2}{3} = \frac{1-2}{3} = -\frac{1}{3}$$

$$\boxed{-\frac{1}{3}}$$

6. Let $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$. Find the linearization of f at $(3, 2)$ and use it to approximate $f(2.99, 1.98)$.

$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$

$f(3,2) = 8$

$f_x(x,y) = 2x - y, \quad f_x(3,2) = 4$

$f_y(x,y) = -x + y, \quad f_y(3,2) = -1$

$$\boxed{L(x,y) = 8 + 4(x-3) - (y-2), \quad L(2.99, 1.98) = 7.98}$$

7. Find and classify the critical points of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

$f_x(x,y) = -6x + 6y = 0 \Rightarrow 6x = 6y$

$f_y(x,y) = 6y - 6y^2 + 6x = 0$

$12y - 6y^2 = 0$

$6y(2-y) = 0$

$y=0,\ y=2$

$x=0,\ x=2$

TWO CRIT PTS $(0,0)$ $(2,2)$

$$d(x,y) = \begin{vmatrix} -6 & 6 \\ 6 & 6-12y \end{vmatrix} = 72y - 72$$

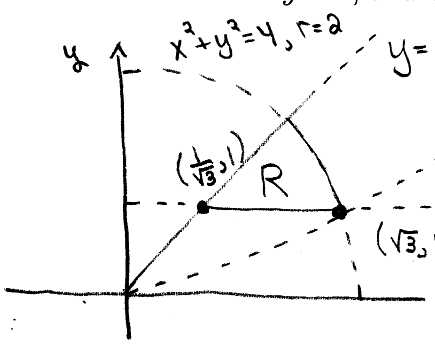
$d(0,0) = -72 \rightarrow$ SADDLE PT

$d(2,2) = 72,\ f_{xx}(2,2) < 0$

$\rightarrow$ REL MAX

$$\boxed{(0,0,0) \text{ IS A SADDLE PT.} \qquad f(2,2) = 8 \text{ IS A REL. MAX.}}$$

8. Use a double integral to find the area of the plane region inside the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$. (Hint: Use polar coordinates.)



$r\sin\theta = 1$

$r = \csc\theta$

$$\int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

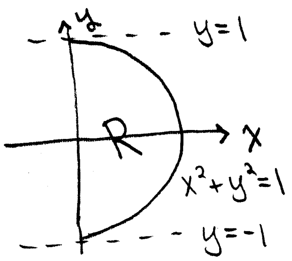
$$= \int_{\pi/6}^{\pi/3} \left(2 - \frac{1}{2}\csc^2\theta\right) d\theta$$

$$= 2\theta + \frac{1}{2}\cot\theta \Big|_{\pi/6}^{\pi/3} = \left(\frac{2\pi}{3} + \frac{1}{2\sqrt{3}}\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

$$\boxed{\frac{\pi}{3} + \frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2} \approx 0.46985}$$

9. Evaluate the iterated integral by first converting to cylindrical coordinates.

$$\underbrace{\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}}_{R}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$$



$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1}\int_{z=0}^{z=r\cos\theta} r^3\,dz\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta$$

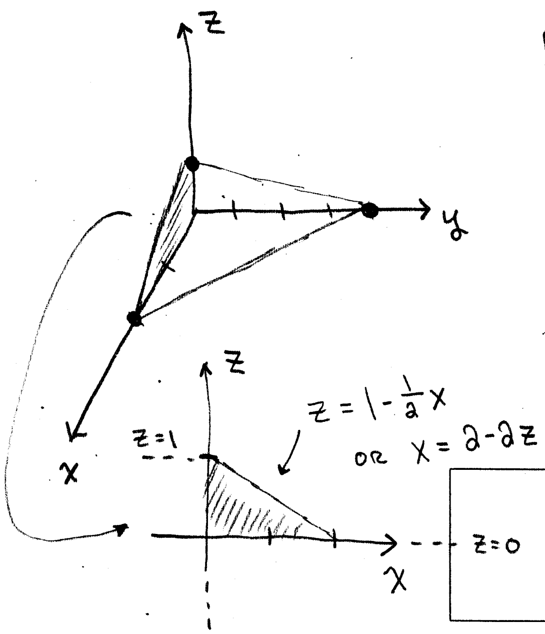
$$= \int_{-\pi/2}^{\pi/2}\frac{1}{5}\cos\theta\,d\theta = \frac{1}{5}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \frac{2}{5}$$

$$\boxed{\frac{2}{5}}$$

10. A tetrahedron lies in the 1st octant bounded by the planes $x=0$, $y=0$, $z=0$, and $2x+y+4z=4$. Its density at the point (x,y,z) is given by $\rho(x,y,z)=1+x+y^2+z^3$. Set up the iterated integral that gives the mass of the solid. Use technology to evaluate the integral.

$$\text{Mass} = \int_{z=0}^{z=1}\int_{x=0}^{x=2-2z}\int_{y=0}^{y=4-4z-2x}(1+x+y^2+z^3)\,dy\,dx\,dz$$

$$= \frac{21}{5}$$



$$\boxed{\int_0^1\int_0^{2-2z}\int_0^{4-4z-2x}(1+x+y^2+z^3)\,dy\,dx\,dz = \frac{21}{5}}$$

Math 233 - Final Exam B

December 16, 2021

Name key

Score

Show all work to receive full credit. Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation. Place your final answer in the box provided.

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1. Determine the measure of angle A in $\triangle ABC$, where

$$A(1,1,8), \quad B(4,-3,-4), \quad C(-3,1,5).$$

Write your answer in degrees rounded to the nearest hundredth.

$\vec{AB} = 3\hat{i} - 4\hat{j} - 12\hat{k}$

$\vec{AC} = -4\hat{i} - 3\hat{k}$

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{\|\vec{AB}\|\,\|\vec{AC}\|} = \frac{-12+36}{\sqrt{169}\,\sqrt{25}} = \frac{24}{65}.$$

$\theta \approx 68.33^\circ$

$$\boxed{68.33^\circ}$$

2. Find a set of symmetric equations for the line passing through the points $(1,-2,3)$ and $(-3,1,1)$.

DIRECTION $= -4\hat{i} + 3\hat{j} - 2\hat{k}$

POINT $(1,-2,3)$

$$\boxed{\frac{x-1}{-4} = \frac{y+2}{3} = \frac{z-3}{-2}}$$

3. Let $\vec{r}(t) = 3\cos(4t)\,\hat{\imath} + 3\sin(4t)\,\hat{\jmath} + 5t\,\hat{k}$. Find the unit tangent vector at the point where $t = \pi/2$.

$$\vec{r}\,'(t) = -12\sin 4t\ \hat{\imath} + 12\cos 4t\ \hat{\jmath} + 5\hat{k}$$

$$\vec{r}\,'\left(\tfrac{\pi}{2}\right) = 0\hat{\imath} + 12\hat{\jmath} + 5\hat{k}$$

$$\left\|\vec{r}\,'\left(\tfrac{\pi}{2}\right)\right\| = \sqrt{144+25} = \sqrt{169} = 13$$

$$\boxed{\hat{T}(t) = \frac{1}{13}\left(12\hat{\jmath} + 5\hat{k}\right)}$$

4. Find the limit or show that it does not exist: $\displaystyle\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$

% More work.

Along $x=2$: $\displaystyle\lim_{y\to1} \frac{0}{(y-1)^2} = 0$

Along $y=1$: $\displaystyle\lim_{x\to1} \frac{0}{(x-2)^2} = 0$

Along $x = 2y$ $\displaystyle\lim_{y\to1} \frac{(2y-2)(y-1)}{(2y-2)^2+(y-1)^2} = \lim_{y\to1} \frac{2(y-1)^2}{5(y-1)^2} = \frac{2}{5}.$

$\boxed{\text{Limit DNE by two-path test}}$

5. Use differentials to approximate the change in $z = \sqrt{41 - 4x^2 - y^2}$ as (x, y) moves from the point $(2, 3)$ to the point $(2.1, 2.9)$.

$$\Delta z \approx \frac{1}{2}\left(41-4x^2-y^2\right)^{-1/2}(-8x)\cdot \Delta x + \frac{1}{2}\left(41-4x^2-y^2\right)^{-1/2}(-2y)\,\Delta y$$

Plug in $x = 2$, $y = 3$, $\Delta x = 0.1$, $\Delta y = -0.1$

$$\Delta z \approx \frac{-8}{\sqrt{16}}(0.1) + \frac{-3}{\sqrt{16}}(-0.1) = \frac{-0.5}{4} = -\frac{1}{8}$$

$$\boxed{-\frac{1}{8} = -0.125}$$

6. Find an equation of the plane tangent to graph of $z = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at the point where $(x, y) = (2, -1)$. $\Rightarrow z = 34$

$$F(x,y,z) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4 - z$$

$$\vec{\nabla} F(x,y,z) = (4x - 3y + 2)\hat{\imath} + (-3x + 16y - 4)\hat{\jmath} - \hat{k}$$

$$\vec{n} = \vec{\nabla} F(2,-1,34) = 13\hat{\imath} - 26\hat{\jmath} - \hat{k}$$

Point $(2,-1,34)$

$$13(x-2) - 26(y+1) - (z-34) = 0$$

or

$$13x - 26y - z = 18$$

$$\boxed{13x - 26y - z = 18}$$

7. Find the directional derivative of $f(x,y,z) = \dfrac{y}{x+z}$ at $P(2,1,-1)$ in the direction from P to $Q(-1,2,0)$.

$$\vec{\nabla} f(x,y,z) = -\frac{y}{(x+z)^2}\hat{i} + \frac{1}{x+z}\hat{j} - \frac{y}{(x+z)^2}\hat{k}$$

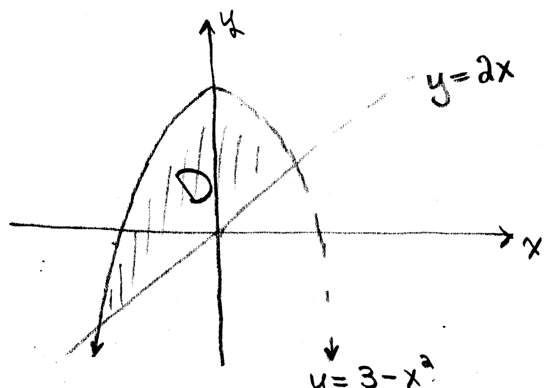
$$\vec{\nabla} f(2,1,-1) = -\hat{i} + \hat{j} - \hat{k}.$$

DIRECTION $= \vec{PQ} = -3\hat{i} + \hat{j} + \hat{k} \qquad \|\vec{PQ}\| = \sqrt{11}$

$$\frac{\vec{\nabla} f(2,1,-1) \cdot \vec{PQ}}{\|\vec{PQ}\|} = \frac{1}{\sqrt{11}}(3+1-1) = \frac{3}{\sqrt{11}}$$

$$\boxed{\frac{3}{\sqrt{11}}}$$

8. Let D be the bounded region between the graphs of $y = 3 - x^2$ and $y = 2x$. Write the double integral as an iterated integral and evaluate.



$$\iint_D x^2\, dA = \int_{x=-3}^{x=1} \int_{y=2x}^{y=3-x^2} x^2\, dy\, dx =$$

$$\int_{-3}^{1} (3x^2 - x^4 - 2x^3)\, dx = x^3 - \frac{1}{5}x^5 - \frac{1}{2}x^4 \Big|_{-3}^{1}$$

$$= \left(1 - \frac{1}{5} - \frac{1}{2}\right) - \left(-27 + \frac{243}{5} - \frac{81}{2}\right)$$

$$= \frac{96}{5}$$

$3 - x^2 = 2x$

$x^2 + 2x - 3 = 0$

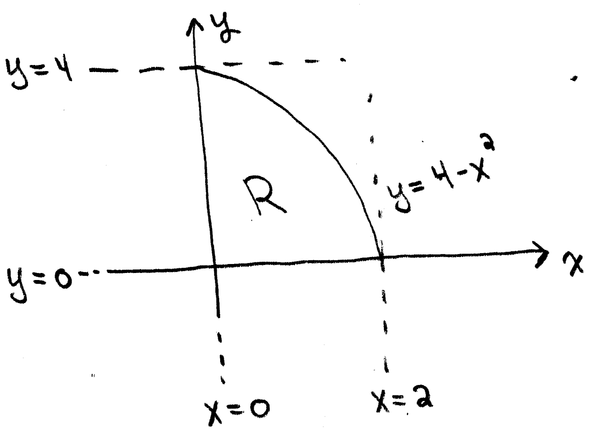
$(x+3)(x-1) = 0$

$x = -3,\ x = 1$

$$\boxed{\frac{96}{5} = 19.2}$$

9. Sketch the region of integration. Then write the new iterated integral with the reversed order of integration. Do not evaluate the integral.

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx$$



$$\int_{y=0}^{y=4} \int_{x=0}^{x=\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

$$\boxed{\int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy}$$

10. Evaluate the line integral $\int_C (x+y+z)\, ds$, where C is the circle described by $\vec{r}(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{k}$ for $0 \leq t \leq 2\pi$.

$\vec{v}(t) = -2\sin t\,\hat{\imath} + 2\cos t\,\hat{k}$

$\|\vec{v}(t)\| = 2$

$$\int_0^{2\pi} (2\cos t + 0 + 2\sin t)(2)\, dt = 0$$

$$\boxed{0}$$