Math 233 - Quiz 10

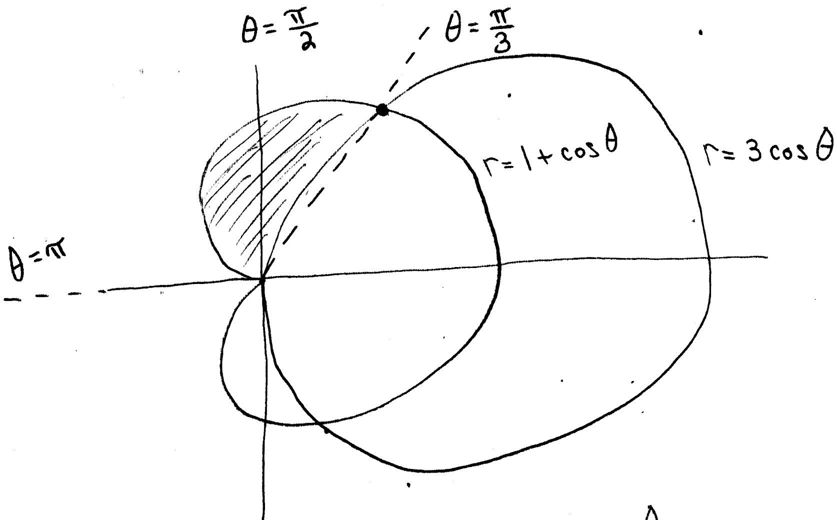
December 1, 2022

Name key

Score

Show all work to receive full credit. Supply explanations when necessary. This quiz is due December 6.

1. (4 points) Use double integrals to find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.



$$1 + \cos\theta = 3\cos\theta$$
$$\Downarrow$$
$$\cos\theta = \frac{1}{2}$$
$$\Downarrow$$
$$\theta = \frac{\pi}{3}$$

$$\text{Area} = 2\left[\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\frac{\pi}{2}}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta\right]$$

$$= 2\left[\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_{3\cos\theta}^{1+\cos\theta} d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}r^2\Big|_{0}^{1+\cos\theta}\right.$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(1 + 2\cos\theta + \cos^2\theta - 9\cos^2\theta\right) d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(2\cos\theta - 3 - 4\cos 2\theta\right) d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

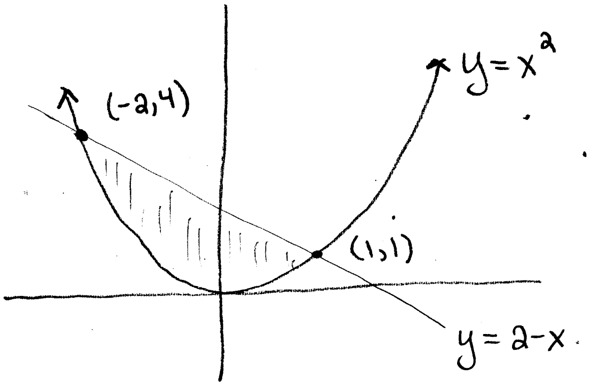
$$= \left(2\sin\theta - 3\theta - 2\sin 2\theta\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_{\frac{\pi}{2}}^{\pi} =$$

Turn over.

$$\left(2 - \frac{\pi}{2}\right) + \left(\frac{3\pi}{4} - 2\right) = \boxed{\frac{\pi}{4}}$$

$$x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$

2. (3 points) A thin plate is bounded by the graphs of $y = x^2$ and $y = 2 - x$. The density of the plate at the point (x, y) is given by $\rho(x, y) = 5 + xy$. Set up the iterated integrals that are required to determine the center of mass of the plate. Then tell how the values of those iterated integrals would be used to calculate the center of mass. Do not evaluate the integrals.



$$m = \int_{-2}^{1} \int_{x^2}^{2-x} (5 + xy)\, dy\, dx$$

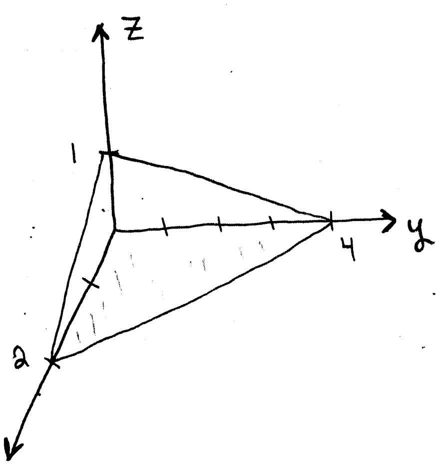
$$M_y = \int_{-2}^{1} \int_{x^2}^{2-x} x(5 + xy)\, dy\, dx$$

$$M_x = \int_{-2}^{1} \int_{x^2}^{2-x} y(5 + xy)\, dy\, dx$$

CENTER OF MASS

$$= \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

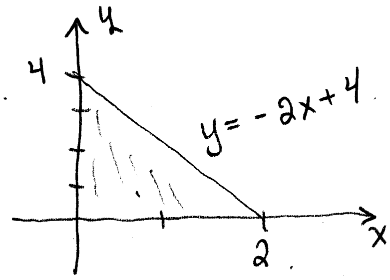
3. (3 points) Use a triple integral to find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + 4z = 4$.



$$\text{VOLUME} = \iiint_T dV$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=-2x+4} \int_{z=0}^{z=\frac{1}{4}(4-y-2x)} 1\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{-2x+4} \frac{1}{4}(4 - y - 2x)\, dy\, dx$$



$$= \frac{1}{4} \int_0^2 \left[4(-2x+4) - \frac{1}{2}(4x^2 - 16x + 16) - 2x(-2x+4) \right] dx$$

$$= \frac{1}{4} \int_0^2 (2x^2 - 8x + 8)\, dx = \frac{1}{6}(2)^3 - (2)^2 + 2(2) = \frac{8}{6}$$

$$= \boxed{\frac{4}{3}}$$