Math 233 - Quiz 7
October 27, 2022

Name key

Score ______

Show all work to receive full credit. Supply explanations when necessary. This quiz is due November 1.

1. (3 points) Use the ϵ_1-ϵ_2 definition of differentiable to show that $f(x,y) = 2x^2 - 5xy + 4y^2$ is differentiable everywhere.

$$z = 2x^2 - 5xy + 4y^2, \quad \frac{\partial z}{\partial x} = 4x - 5y, \quad \frac{\partial z}{\partial y} = -5x + 8y$$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \left[2(x+\Delta x)^2 - 5(x+\Delta x)(y+\Delta y) + 4(y+\Delta y)^2\right] - \left[2x^2 - 5xy + 4y^2\right]$$

$$= \cancel{2x^2} + \underline{4x\Delta x} + \underline{\underline{2\Delta x^2}} - \cancel{5xy} - \underline{5x\Delta y} - \underline{5y\Delta x} - \underline{5\Delta x\Delta y} + \cancel{4y^2} + \underline{8y\Delta y} + 4\Delta y^2 - \cancel{2x^2} + \cancel{5xy} - \cancel{4y^2}$$

$$= (4x - 5y)\Delta x + (-5x + 8y)\Delta y + (2\Delta x - 5\Delta y)\Delta x + (4\Delta y)\Delta y$$

$$= \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

WHERE $\epsilon_1 = 2\Delta x - 5\Delta y$ AND $\epsilon_2 = 4\Delta y$.

THIS FORMULA FOR Δz HOLDS FOR ALL (x,y) IN $\mathbb{R}^2$.

FURTHERMORE,

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \epsilon_1 = 0 \quad \text{AND} \quad \lim_{(\Delta x, \Delta y)\to(0,0)} \epsilon_2 = 0.$$

CONCLUSION: f IS DIFFERENTIABLE ON $\mathbb{R}^2$.

Turn over.

2. (2 points) Let $f(x, y) = x^2 \sin(2y)$. Find an equation of the plane tangent to the graph of f at the point $(2, \pi/6)$.

$f_x(x,y) = 2x \sin(2y)$, $f_x(2, \pi/6) = 4 \sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

$f_y(x,y) = 2x^2 \cos(2y)$, $f_y(2, \pi/6) = 8 \cos\left(\frac{\pi}{3}\right) = 4$

$f(2, \pi/6) = 4 \sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

TAN. PLANE:

$z = 2\sqrt{3} + 2\sqrt{3}(x-2) + 4\left(y - \frac{\pi}{6}\right)$ OR $2\sqrt{3}(x-2) + 4\left(y-\frac{\pi}{6}\right) - (z - 2\sqrt{3}) = 0$.

3. (2 points) Find the linearization of $g(x, y) = \tan^{-1}(xy^2)$ at the point $(1, 1)$. Then use your linearization to estimate $g(1.02, 0.97)$.

$g_x(x,y) = \dfrac{y^2}{1 + (xy^2)^2}$; $g_x(1,1) = \dfrac{1}{2}$

$g_y(x,y) = \dfrac{2xy}{1 + (xy^2)^2}$; $g_y(1,1) = 1$

$g(1,1) = \tan^{-1}(1) = \dfrac{\pi}{4}$

$L(x,y) = \frac{\pi}{4} + \frac{1}{2}(x-1) + (y-1)$

$g(1.02, 0.97) \approx L(1.02, 0.97)$

$= \frac{\pi}{4} + \frac{1}{2}(0.02) + (-0.03)$

$= \frac{\pi}{4} - 0.02 \approx 0.7654$

4. (3 points) Suppose that $w = 3xy + yz$ and that x, y, and z are functions of u and v such that

$$x = \ln u + \cos v, \qquad y = 1 + u \sin v, \qquad z = uv.$$

Use the appropriate chain rule to find $\partial w/\partial u$ at $(u, v) = (1, \pi)$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (3y)\left(\frac{1}{u}\right) + (3x + z)(\sin v) + (y)(v)$$

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(1,\pi)} = 3(1) + (-3+\pi)(0) + (1)(\pi) = \boxed{3 + \pi}$$