Math 233 - Test 1

September 15, 2022

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (9 points) In the following problems, the vectors $\vec{u}$ and $\vec{w}$ are 2D vectors in the xy-plane.

   (a) The vector $\vec{w}$ has magnitude 5 and makes a 120° angle with the positive x-axis. Find the component form of $\vec{w}$.

   $$\vec{w} = 5\cos 120°\,\hat{\imath} + 5\sin 120°\,\hat{\jmath}$$

   $$= 5\left(-\tfrac{1}{2}\right)\hat{\imath} + 5\left(\tfrac{\sqrt{3}}{2}\right)\hat{\jmath} = \boxed{-\frac{5}{2}\hat{\imath} + \frac{5\sqrt{3}}{2}\hat{\jmath}}$$

   (b) The vector $\vec{u} + \vec{w}$ has component form $-3\,\hat{\imath} + \sqrt{3}\,\hat{\jmath}$. Find the component form of $\vec{u}$.

   $$\vec{u} + \left(-\frac{5}{2}\hat{\imath} + \frac{5\sqrt{3}}{2}\hat{\jmath}\right) = -3\hat{\imath} + \sqrt{3}\hat{\jmath}$$

   $$\Rightarrow \vec{u} = \left(-3 + \frac{5}{2}\right)\hat{\imath} + \left(\sqrt{3} - \frac{5\sqrt{3}}{2}\right)\hat{\jmath}$$

   $$\Rightarrow \boxed{\vec{u} = -\frac{1}{2}\hat{\imath} - \frac{3\sqrt{3}}{2}\hat{\jmath}}$$

   (c) What angle does $\vec{u}$ make with the positive x-axis?

   $$\|\vec{u}\| = \sqrt{\frac{1}{4} + \frac{27}{4}} = \sqrt{\frac{28}{4}} = \sqrt{7}$$

   $\cos\alpha = \dfrac{-1/2}{\sqrt{7}}$ AND α IS IN QUAD III $\Rightarrow \boxed{\alpha \approx 259.1°}$

2. (5 points) Find a vector of magnitude 5 that is parallel to the line with symmetric equations

   $$\frac{5-x}{3} = \frac{y-6}{4} = z.$$

   $$\frac{x-5}{-3} = \frac{y-6}{4} = \frac{z-0}{1}$$

   $$\vec{v} = -3\hat{\imath} + 4\hat{\jmath} + \hat{k}$$

   $$\|\vec{v}\| = \sqrt{9+16+1} = \sqrt{26}$$

   $$\boxed{\frac{5\vec{v}}{\|\vec{v}\|} = \frac{-15}{\sqrt{26}}\hat{\imath} + \frac{20}{\sqrt{26}}\hat{\jmath} + \frac{5}{\sqrt{26}}\hat{k}}$$

3. (8 points) Find the acute angle between the planes described by the equations below. Write your final answer in degrees, rounded to the nearest hundredth.

$$3x + 4y - 9z = 1 \qquad\qquad 2y + 8z = 5$$

$\vec{n}_1 = 3\hat{i} + 4\hat{j} - 9\hat{k}$

$\vec{n}_2 = 0\hat{i} + 2\hat{j} + 8\hat{k}$

$|\vec{n}_1 \cdot \vec{n}_2| = |0 + 8 - 72|$
$= |-64|$
$= 64$

$\|\vec{n}_1\| = \sqrt{9+16+81}$
$= \sqrt{106}$

$\|\vec{n}_2\| = \sqrt{4+64}$
$= \sqrt{68}$

$\cos\theta = \dfrac{64}{\sqrt{106}\sqrt{68}}$

$\Downarrow$

$\boxed{\theta \approx 41.08°}$

4. (12 points) Consider the plane described by the equation $2x - y + 6z = 7$.

(a) Find a point on the plane. Call it P, and show (or explain) how you know P is on the plane.

(1,1,1) IS A POINT ON THE PLANE BECAUSE IT SATISFIES THE EQUATION: $2(1) - (1) + 6(1) = 2 - 1 + 6 = 7$. ✓

(b) Let $Q\,(4, -3, 2)$. Show that Q is NOT on the plane.

$2(4) - (-3) + 6(2) = 8 + 3 + 12 = 23 \neq 7$

Q DOES NOT SATISFY THE EQUATION.

(c) Let $\vec{n}$ be a vector normal to the plane. Compute $\text{proj}_{\vec{n}} \vec{PQ}$.

$\vec{n} = 2\hat{i} - \hat{j} + 6\hat{k}$

$\vec{PQ} = 3\hat{i} - 4\hat{j} + \hat{k}$

$\vec{n} \cdot \vec{PQ} = 6 + 4 + 6 = 16$

$\vec{n} \cdot \vec{n} = 4 + 1 + 36 = 41$

$\boxed{\text{proj}_{\vec{n}} \vec{PQ} = \frac{16}{41}(2\hat{i} - \hat{j} + 6\hat{k})}$

(d) Compute $\|\text{proj}_{\vec{n}} \vec{PQ}\|$. (You have just computed the distance from Q to the plane.)

$\left\| \frac{16}{41}(2\hat{i} - \hat{j} + 6\hat{k}) \right\| = \frac{16}{41}\sqrt{41} = \boxed{\frac{16}{\sqrt{41}} \approx 2.5}$

5. (10 points) A triangle has vertices at the points $A(1,3,-2)$, $B(1,1,-5)$, and $C(8,0,-3)$.

(a) Find the area of $\triangle ABC$.

$\vec{AB} = 0\hat{i} - 2\hat{j} - 3\hat{k}$

$\vec{AC} = 7\hat{i} - 3\hat{j} - \hat{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -3 \\ 7 & -3 & -1 \end{vmatrix} = \hat{i}(-7) - \hat{j}(21) + \hat{k}(14)$$

$$= -7\hat{i} - 21\hat{j} + 14\hat{k}$$

$$\text{Area} = \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{49 + 441 + 196} = \boxed{\frac{\sqrt{686}}{2} \approx 13.1}$$

(b) Find an equation of the plane containing A, B, and C.

From $\vec{AB} \times \vec{AC}$,

I'll use

$\vec{n} = \hat{i} + 3\hat{j} - 2\hat{k}$

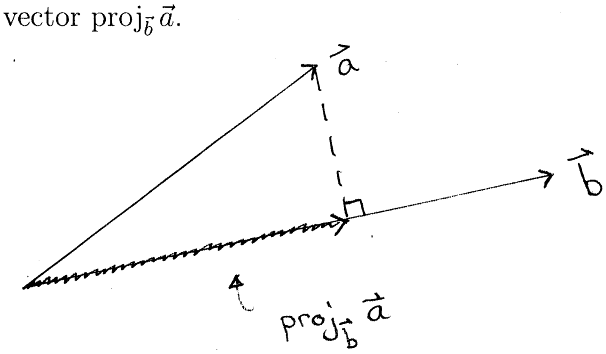
$1x + 3y - 2z = d$

Using $A(1,3,-2)$,

$d = 1 + 9 + 4 = 14$

$$\boxed{x + 3y - 2z = 14}$$

6. (3 points) Sketch two vectors that share a common initial point. Label them $\vec{a}$ and $\vec{b}$. Then sketch and label the vector $\text{proj}_{\vec{b}}\,\vec{a}$.



7. (6 points) Show that the points are collinear. Explain your reasoning.

$P(2,-1,7)$, $Q(17,-10,10)$, $R(-8,5,5)$

$\vec{PQ} = 15\hat{i} - 9\hat{j} + 3\hat{k}$

$\vec{PR} = -10\hat{i} + 6\hat{j} - 2\hat{k}$

$\Rightarrow \vec{PR} = -\frac{2}{3}\vec{PQ}$

$\Rightarrow$ $\vec{PR}$ IS PARALLEL TO $\vec{PQ}$

$\Rightarrow$ SINCE P IS A SHARED POINT,

P, Q, R MUST BE COLLINEAR.

8. (6 points) What does it mean for two vectors to be orthogonal? Give an example of two nonzero, orthogonal vectors in 3D-space, and show that your vectors are orthogonal.

$\vec{u}$ AND $\vec{v}$ ARE ORTHOGONAL IF $\vec{u} \cdot \vec{v} = 0$.

LET $\vec{u} = \hat{i} - 5\hat{j} + 3\hat{k}$

AND $\vec{v} = 6\hat{i} + 0\hat{j} - 2\hat{k}$.

$\vec{u} \cdot \vec{v} = (1)(6) + (-5)(0) + (3)(-2)$

$= 6 + 0 - 6 = 0$

$\vec{u}$ IS ORTHOG. TO $\vec{v}$!

9. (6 points) Find a set of parametric equations for the line **segment** connecting $R(3, -4, -2)$ and $S(4, 3, 9)$.

$\vec{RS} = \hat{i} + 7\hat{j} + 11\hat{k}$

USING $S(4,3,9)$ ....

$$\boxed{\begin{aligned} x &= t + 4 \\ y &= 7t + 3 \\ z &= 11t + 9 \end{aligned} \qquad -1 \le t \le 0}$$

10. (4 points) Suppose you were given two nonzero vectors $\vec{u}$ and $\vec{v}$ in 3D-space. Explain how you could find a nonzero vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v}$ IS ORTHOGONAL TO BOTH $\vec{u}$ & $\vec{v}$.

IF $\vec{u}$ AND $\vec{v}$ ARE PARALLEL, THE $\vec{u} \times \vec{v} = \vec{0}$.

IN THIS CASE, FIND ANY VECTOR ORTHOG. TO $\vec{u}$, AND IT WILL ALSO BE ORTHOG. TO $\vec{v}$.

11. (6 points) Suppose you were given two nonzero vectors in 3D-space. Briefly describe two different ways that you could test whether the vectors are parallel.

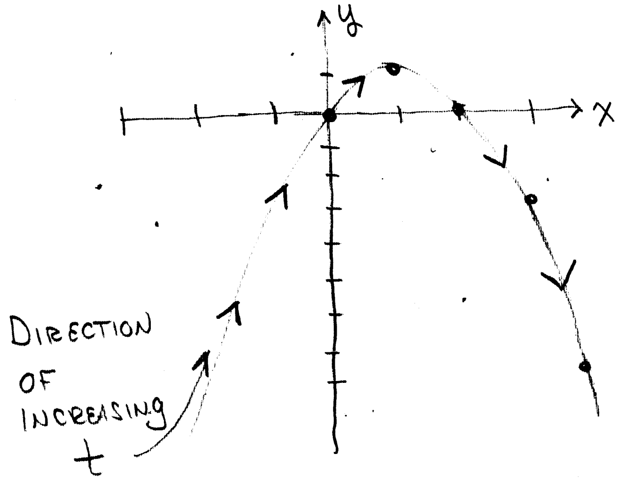
1. THEY ARE PARALLEL IF, IN COMPONENT FORM, ONE IS A SCALAR MULTIPLE OF THE OTHER.

2. THEY ARE PARALLEL IF THEIR CROSS PRODUCT IS THE ZERO VECTOR.

$$x = 1+t$$
$$y = 1-t^2 \Rightarrow y = 1-(x-1)^2 = 1-x^2+2x-1$$
$$= 2x-x^2$$

12. (8 points) Plot four or five points on the graph of $\vec{r}(t) = (1+t)\,\hat{\imath} + (1-t^2)\,\hat{\jmath}$. Then sketch the graph. Draw arrows on your graph to indicate the curve's orientation.

t	$(x(t), y(t))$
-1	$(0,0)$
0	$(1,1)$
1	$(2,0)$
2	$(3,-3)$
3	$(4,-8)$



$y = 2x - x^2$

13. (3 points) Look back at the problem above. Now describe the graph of $\vec{R}(t) = (1+t)\,\hat{\imath} + (1-t^2)\,\hat{\jmath} + t\,\hat{k}$.

THE GRAPH OF $\vec{R}$ IS IN 3-SPACE. THE Z-COORD OF EACH PT IS t, SO THE GRAPH IS BEHIND THE SHEET FOR NEG t AND COMING OUT OF THE SHEET FOR POS t. BUT THE PROJECTION ONTO THE XY-PLANE IS THE SAME GRAPH AS ABOVE.

14. (6 points) Let $\vec{r}(t) = \sqrt{t-1}\,\hat{\imath} + \dfrac{\ln t}{t-5}\,\hat{\jmath} + \cos(\pi t)\,\hat{k}$.

(a) Determine the domain of $\vec{r}$.

$\sqrt{t-1}$: $t \geq 1$

$\dfrac{\ln t}{t-5}$: $t > 0,\ t \neq 5$

$\cos(\pi t)$: $\mathbb{R}$

DOMAIN IS

$\{t : t \geq 1 \ \&\ t \neq 5\}$

(b) Compute $\lim_{t \to 17} \vec{r}(t)$.

$$\lim_{t \to 17} \vec{r}(t) = \sqrt{16}\,\hat{\imath} + \frac{\ln 17}{12}\,\hat{\jmath} + \cos(17\pi)\,\hat{k}$$

$$= \boxed{4\hat{\imath} + \frac{\ln 17}{12}\,\hat{\jmath} - \hat{k}}$$

15. (8 points) Let $\vec{r}(t) = \sin 6t\,\hat{\imath} + \cos 6t\,\hat{\jmath} + 8t\,\hat{k}$.

(a) Let $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$. Compute $\hat{T}(t)$.

$$\vec{r}\,'(t) = 6\cos 6t\,\hat{\imath} - 6\sin 6t\,\hat{\jmath} + 8\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{36\cos^2 6t + 36\sin^2 6t + 64} = \sqrt{36+64}$$

$$= \sqrt{100} = 10$$

$$\boxed{\hat{T}(t) = \frac{3}{5}\cos 6t\,\hat{\imath} - \frac{3}{5}\sin 6t\,\hat{\jmath} + \frac{4}{5}\hat{k}}$$

(b) Compute $\hat{T}(t) \cdot \hat{T}\,'(t)$.

$$\hat{T}\,'(t) = -\frac{18}{5}\sin 6t\,\hat{\imath} - \frac{18}{5}\cos 6t\,\hat{\jmath} + 0\hat{k}$$

$$\boxed{\hat{T}(t) \cdot \hat{T}\,'(t) = 0}$$

DON'T EVEN NEED TO COMPUTE IT.

A VECTOR OF CONSTANT MAG. IS ORTHOG.

TO ITS DERIVATIVE.