Math 233 - Test 2
October 13, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Find the function $\vec{r}(t)$ that satisfies

$$\vec{r}\,'(t) = e^{-2t}\,\hat{\imath} - t\,\hat{\jmath} + \frac{5t}{t^2+1}\,\hat{k}; \qquad \vec{r}(0) = 5\hat{\imath} + 3\hat{\jmath} - 2\hat{k}.$$

$x(t) = \int e^{-2t}\,dt = -\frac{1}{2}e^{-2t} + c_1 \qquad x(0) = -\frac{1}{2} + c_1 = 5 \Rightarrow c_1 = \frac{11}{2}$

$y(t) = \int -t\,dt = -\frac{1}{2}t^2 + c_2 \qquad y(0) = c_2 = 3$

$z(t) = \int \frac{5t}{t^2+1}\,dt = \frac{5}{2}\int \frac{1}{u}\,du \qquad z(0) = c_3 = -2$

$u = t^2+1$

$du = 2t\,dt$

$= \frac{5}{2}\ln(t^2+1) + c_3$

$$\boxed{\vec{r}(t) = \left(-\frac{1}{2}e^{-2t} + \frac{11}{2}\right)\hat{\imath} + \left(-\frac{1}{2}t^2 + 3\right)\hat{\jmath} + \left[\frac{5}{2}\ln(t^2+1) - 2\right]\hat{k}}$$

2. (10 points) Let $\vec{r}(t) = (t^2 - t)\,\hat{\imath} + \frac{1}{6}(4t-1)^{3/2}\,\hat{\jmath} + 5\,\hat{k}$. Starting from $t = 1$, reparameterize $\vec{r}$ in terms of the arc-length parameter s.

$\vec{r}\,'(t) = (2t-1)\hat{\imath} + \frac{1}{6}\left(\frac{3}{2}\right)(4t-1)^{1/2}(4)\hat{\jmath}$

$= (2t-1)\hat{\imath} + \sqrt{4t-1}\,\hat{\jmath}$

$\|\vec{r}\,'(t)\| = \sqrt{(2t-1)^2 + (4t-1)}$

$= \sqrt{4t^2} = 2t$

$s = \int_1^t 2u\,du = u^2\Big|_1^t = t^2 - 1$

$s = t^2 - 1 \Rightarrow t = \sqrt{s+1}, \quad s \geq 0$

$$\boxed{\vec{R}(s) = \left(s+1-\sqrt{s+1}\right)\hat{\imath} + \frac{1}{6}\left(4\sqrt{s+1} - 1\right)^{3/2}\hat{\jmath} + 5\hat{k}}$$

3. (8 points) For $-\pi/2 < x < \pi/2$, let $f(x) = \ln(\cos x)$. Compute the curvature function and say where the graph of f has its maximum curvature.

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$f''(x) = -\sec^2 x$$

$$\sqrt{1+[f'(x)]^2} = \sqrt{1+\tan^2 x}$$

$$= |\sec x|$$

$$= \sec x \quad \text{FOR} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$K(x) = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}}$$

$$K(x) = \frac{\sec^2 x}{\sec^3 x} = \frac{1}{\sec x}$$

$$K(x) = \cos x.$$

ON $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, THE MAX VALUE OF $K(x) = \cos x$ IS 1 AT $x = 0$.

4. (2 points) An object is moving along a curve in such a way that the normal component of its acceleration is zero. What can you say about the motion of the object?

THE DIRECTION OF THE OBJECT CANNOT BE CHANGING.

THE MOTION IS ALONG A LINE. IN PHYSICS, THIS IS CALLED RECTILINEAR MOTION.

5. (2 points) An object is moving along a curve in such a way that the tangential component of its acceleration is zero. What can you say about the motion of the object?

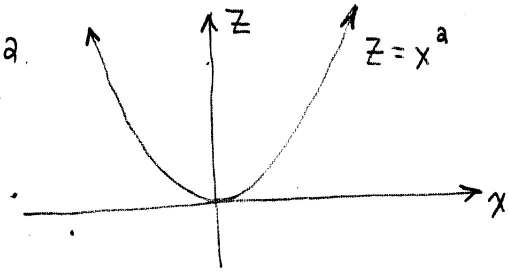
THE OBJECT CANNOT BE CHANGING SPEED.

THE OBJECT IS MOVING AT CONSTANT SPEED, BUT ITS DIRECTION MAY BE CHANGING.

6. (6 points) Consider the surface described by the equation $z - 4y^2 = x^2$.

(a) Fix a value for one of the variables and draw a good sketch of the corresponding level curve.

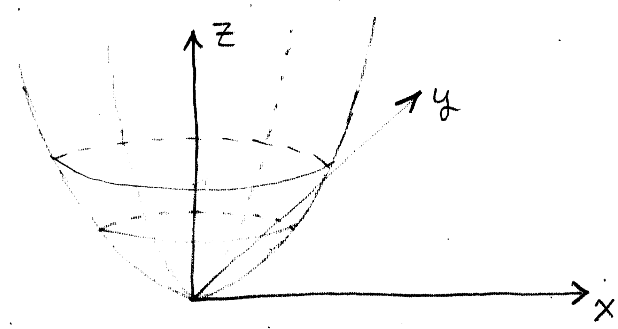
$y = 0 \Rightarrow z = x^2$



(b) Identify the surface.

$z = x^2 + 4y^2$ DESCRIBES AN ELLIPTIC PARABOLOID OPENING UP THE Z-AXIS.

(c) Sketch a rough graph of the surface.



7. (6 points) Consider the function $F(x, y) = \sqrt{y^2 - 2x^2}$.

(a) What is the domain of F?

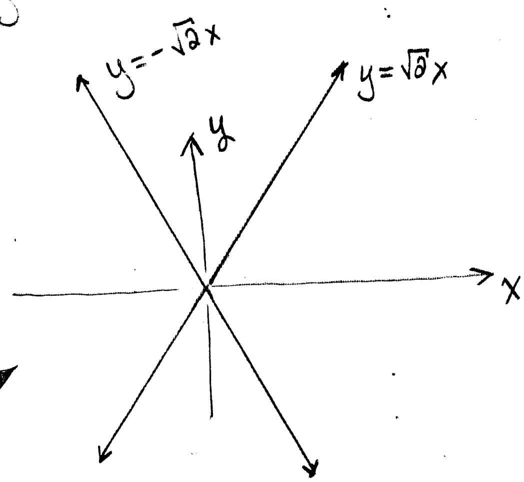
$\{(x,y) : y^2 \geq 2x^2\}$

(b) Sketch the level curve $F(x, y) = 0$.

$F(x,y) = 0 \Rightarrow y^2 = 2x^2$

$\Rightarrow y = \pm\sqrt{2}\,x$

A PAIR OF LINES.



(c) Describe the graph of F.

$z^2 = y^2 - 2x^2,\ z \geq 0 \Rightarrow 2x^2 + z^2 = y^2,\ z \geq 0$

UPPER HALF ($z \geq 0$) OF AN ELLIPTIC CONE OPENING UP/DOWN THE Y-AXIS.

$$\vec{r}(t) = 80\cos(-60°)\,t\ \hat{\imath} + \left(-16t^2 + 80\sin(-60)t + 168\right)\hat{\jmath}$$
$$= 40t\ \hat{\imath} + \left(-16t^2 - 40\sqrt{3}\,t + 168\right)\hat{\jmath}$$

8. (20 points) A rock is thrown downward from the top of a building that is 168 ft high at an angle of 60° below the horizontal. The initial speed of the rock is 80 ft/sec. Ignore air resistance and use $g = 32\,\text{ft/sec}^2$.

(a) When will the rock hit the ground? $y(t) = 0$

$$-16t^2 - 40\sqrt{3}\,t + 168 = 0$$

$$\boxed{t = \sqrt{3}\ \text{SEC}}$$

$$\Rightarrow t = \frac{40\sqrt{3} \pm \sqrt{4800 + 4(16)(168)}}{-32} = \frac{40\sqrt{3} \pm 72\sqrt{3}}{-32}$$

WE OBVIOUSLY WANT $\dfrac{-32\sqrt{3}}{-32}$

(b) How far from the base of the building will the rock land?

$x(t) = 40t \Rightarrow 40t$ WHEN $t = \sqrt{3}$

$$\Rightarrow \boxed{40\sqrt{3}\ \text{FT}}$$

(c) What will be the speed of the rock when it hits the ground?

$$\vec{r}\,'(t) = 40\hat{\imath} + \left(-32t - 40\sqrt{3}\right)\hat{\jmath}$$

$$\vec{r}\,'(\sqrt{3}) = 40\hat{\imath} + \left(-72\sqrt{3}\right)\hat{\jmath}$$

$$\|\vec{r}\,'(3)\| = \sqrt{(40)^2 + (72\sqrt{3})^2}$$

$$\approx \boxed{130.97\ \text{FT/SEC}}$$

(d) Set up the definite integral that gives the length of the path of the rock. Use your calculator to estimate the value of your integral.

$$\int_0^{\sqrt{3}} \|\vec{r}\,'(t)\|\,dt = \int_0^{\sqrt{3}} \sqrt{40^2 + \left(-32t - 40\sqrt{3}\right)^2}\,dt$$

$$\approx \boxed{182.04\ \text{FT}}$$

9. (6 points) Describe what it means to be a neighborhood of the point $(0,0)$.

A NEIGHBORHOOD OF $(0,0)$ IS ANY SET THAT ENTIRELY CONTAINS AN OPEN DISK CENTERED AT $(0,0)$.

Follow-up problem: Let D be the set of all points in the circle $x^2 + y^2 = 1$ for which $x \neq y$. More formally,

$$D = \{(x,y) : x^2 + y^2 < 1 \text{ and } x \neq y\}.$$

Explain why D is NOT a neighborhood of $(0,0)$.

ANY OPEN DISK CENTERED AT $(0,0)$ WILL CONTAIN POINTS WHERE $X = Y$. IN OTHER WORD, ANY OPEN DISK CENTERED AT $(0,0)$ CONTAINS POINTS NOT IN D.

10. (12 points) Use the two-path test to show that each limit fails to exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

ALONG $X = 0$ ...

$$\lim_{y\to 0} \frac{0}{y^2} = 0$$

ALONG $y = X$ ...

$$\lim_{X\to 0} \frac{X^2}{2X^2} = \frac{1}{2}$$

TWO DIFF. LIMITS ALONG TWO DIFF. PATHS

$\Rightarrow$ LIMIT DNE

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

ALONG $X = 0$ ...

$$\lim_{y\to 0} \frac{0}{y^2} = 0$$

ALONG $y = X^2$ ...

$$\lim_{X\to 0} \frac{X^4}{2X^4} = \frac{1}{2}$$

TWO DIFF. LIMITS ALONG TWO DIFF. PATHS

$\Rightarrow$ LIMIT DNE

11. (10 points) Compute each limit.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^4 - 16y^2}{x^2 + 4y}$ % MORE WORK.

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2-4y)\cancel{(x^2+4y)}}{\cancel{x^2+4y}} = \lim_{(x,y)\to(0,0)} (x^2-4y)$$

$$= \boxed{0}$$

(b) $\lim_{(x,y)\to(2,2)} \frac{3x - 3y}{\sqrt{x} - \sqrt{y}}$ % MORE WORK.

$$\lim_{(x,y)\to(2,2)} \frac{3x-3y}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(2,2)} \frac{3\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$$

$$= 3(\sqrt{2}+\sqrt{2}) = \boxed{6\sqrt{2}}$$

12. (8 points) Let $\vec{r}(t) = -\cos 3t\,\hat{\imath} - \sin 3t\,\hat{\jmath} + 4t\,\hat{k}$. Compute $\hat{N}(t)$.

$$\vec{r}\,'(t) = 3\sin 3t\,\hat{\imath} - 3\cos 3t\,\hat{\jmath} + 4\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16}$$

$$= \sqrt{25} = 5$$

$$\hat{T}(t) = \frac{3}{5}\sin 3t\,\hat{\imath} - \frac{3}{5}\cos 3t\,\hat{\jmath} + \frac{4}{5}\hat{k}$$

$$\hat{T}'(t) = \frac{9}{5}\cos 3t\,\hat{\imath} + \frac{9}{5}\sin 3t\,\hat{\jmath}$$

$$\|\hat{T}'(t)\| = \frac{9}{5}$$

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$$

$$\hat{N}(t) = \cos 3t\,\hat{\imath} + \sin 3t\,\hat{\jmath}$$