Math 233 - Test 3

November 10, 2022

Name key

Score ________

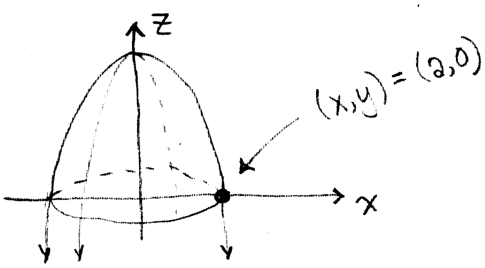
Show all work to receive full credit. Supply explanations where necessary. You must work individually on this test. The test is due at 9:30 am on November 15.

1. (6 points) Let $f(x) = \tan^{-1}(\frac{y}{x})$. Evaluate f_x and f_y at the point $(2, -2)$.

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} \qquad \boxed{f_x(2,-2) = \frac{2}{8} = \frac{1}{4}}$$

$$f_y(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \qquad \boxed{f_y(2,-2) = \frac{2}{8} = \frac{1}{4}}$$

2. (4 points) Think about the graph of the function $f(x, y) = 4 - x^2 - y^2$. Without computing any derivatives, explain how/why we should know that $f_x(2, 0)$ is negative.



THE GRAPH OF THE FUNCTION IS A PARABOLOID OPENING DOWNWARD WITH VERTEX AT (0,0,4).

AT THE POINT WHERE (x,y) = (2,0), THE TANGENT LINE IN THE DIRECTION OF $\hat{i}$ SLANTS DOWNWARD.

3. (4 points) The function f is defined below. It is continuous everywhere, but it can be shown that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. Use our theorem about the equality of mixed partial derivatives to draw a conclusion about f_{xy} or f_{yx}.

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

IF f_{xy} AND f_{yx} WERE CONTINUOUS AT (0,0), THEN THEY WOULD BE EQUAL. THE CONCLUSION IS THAT f_{xy} AND/OR f_{yx} IS NOT CONTINUOUS.

4. (6 points) For $x + y \neq 0$, let $g(x,y,z) = \dfrac{7xz}{x+y}$. Compute g_{xzyz}. Show work or explain your reasoning.

AS LONG AS WE STAY AWAY FROM $X+Y=0$, I EXPECT ALL PARTIAL (INCLUDING MIXED PARTIAL) DERIVATIVES TO BE CONTINUOUS.

MIXED PARTIALS WILL BE EQUAL, SO I'LL COMPUTE g_{zzxy}.

$$g_z = \frac{7x}{x+y}, \quad g_{zz} = 0, \quad \Rightarrow \boxed{g_{zzxy} = 0}$$

5. (8 points) Let $w = x^2yz^3 + \sin(yz)$. Use differentials to estimate the change in w that accompanies the change in (x,y,z) from the point $(2,\pi,1)$ to $(1.9, 3, 1.1)$.

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z = (2xyz^3)\Delta x + (x^2z^3 + z\cos(yz))\Delta y + (3x^2yz^2 + y\cos(yz))\Delta z$$

$$(x,y,z) = (2,\pi,1), \quad (\Delta x, \Delta y, \Delta z) = (-0.1, 3-\pi, 0.1)$$

$$\Delta w \approx (4\pi)(-0.1) + (3)(3-\pi) + (11\pi)(0.1)$$

$$= 9 - 2.3\pi \approx 1.7743$$

6. (10 points) The total resistance R of two resistors connected in parallel satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \qquad R = \frac{R_1R_2}{R_2+R_1}$$

where R_1 and R_2 are the resistances of the connected resistors. Find the linearization of R at $(R_1, R_2) = (10, 15)$ and use it to approximate the total resistance when $R_1 = 10.4$ and $R_2 = 14.7$.

$$\frac{\partial R}{\partial R_1} = \frac{(R_2+R_1)(R_2) - (R_1R_2)(1)}{(R_2+R_1)^2} = \frac{R_2^2}{(R_2+R_1)^2}$$

$$\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_2+R_1)^2}$$

$$R(10,15) = 6$$

$$\left.\frac{\partial R}{\partial R_1}\right|_{(10,15)} = 0.36$$

$$\left.\frac{\partial R}{\partial R_2}\right|_{(10,15)} = 0.16$$

$$L(R_1,R_2) = 6 + 0.36(R_1 - 10) + 0.16(R_2 - 15)$$

$$R(10.4, 14.7) \approx L(10.4, 14.7) = 6 + 0.36(0.4) + 0.16(-0.3) = \boxed{6.096}$$

7. (6 points) Suppose that z is a function of x and y. Further suppose that x and y are converted to polar coordinates by $x = r\cos\theta$ and $y = r\sin\theta$. Write the formulas for $\partial z/\partial r$ and $\partial z/\partial\theta$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \left(\frac{\partial z}{\partial x}\right)(\cos\theta) + \left(\frac{\partial z}{\partial y}\right)(\sin\theta)$$

$$= \boxed{\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right)(-r\sin\theta) + \left(\frac{\partial z}{\partial y}\right)(r\cos\theta)$$

$$= -r\sin\theta\,\frac{\partial z}{\partial x} + r\cos\theta\,\frac{\partial z}{\partial y}$$

8. (6 points) Suppose that y is implicitly defined as a function of x and z by the equation

$$\underbrace{3x^2z - x^2y^2 + 2z^3 + 3yz - 5}_{F(x,y,z)} = 0.$$

Find $\partial y/\partial z$.

$$\frac{\partial y}{\partial z} = \frac{-F_z}{F_y} = \frac{-(3x^2 + 6z^2 + 3y)}{-2x^2y + 3z}$$

9. (6 points) Compute the directional derivative of $g(x, y, z) = xye^z$ at $(2, 4, 0)$ in the direction from $(2, 4, 0)$ to $(0, 0, 0)$.

P Q

$$\vec{PQ} = -2\hat{i} - 4\hat{j}$$

$$\|\vec{PQ}\| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{\nabla} g(x,y,z) = ye^z\,\hat{i} + xe^z\,\hat{j} + xye^z\,\hat{k}$$

$$\vec{\nabla} g(2,4,0) = 4\hat{i} + 2\hat{j} + 8\hat{k}$$

$$\frac{\vec{\nabla} g(2,4,0)\cdot\vec{PQ}}{\|\vec{PQ}\|} = \frac{-16}{2\sqrt{5}} = \frac{-8}{\sqrt{5}}$$

10. (6 points) The temperature at the point (x, y) on a metal plate is given by

$$T = \frac{x}{x^2+y^2}.$$

Find the $\underbrace{\text{direction of greatest increase in heat from}}_{\text{DIRECTION OF THE GRADIENT VECTOR.}}$ the point $(3, 4)$.

$$\vec{\nabla} T(x,y) = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2}\,\hat{\imath} + \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}\,\hat{\imath} + \frac{-2xy}{(x^2+y^2)^2}\,\hat{\jmath}$$

$$\vec{\nabla} T(3,4) = \frac{7}{25^2}\,\hat{\imath} + \frac{-24}{25^2}\,\hat{\jmath}$$

$$= \frac{1}{625}(7\hat{\imath} - 24\hat{\jmath})$$

$$\boxed{\frac{\vec{\nabla} T(3,4)}{\|\vec{\nabla} T(3,4)\|} = \frac{7}{25}\,\hat{\imath} - \frac{24}{25}\,\hat{\jmath}}$$

$$\sqrt{7^2+(24)^2} = 25$$

11. (10 points) Find the point(s) on the surface $z = xy + \frac{1}{x} + \frac{1}{y}$ at which the tangent plane is horizontal.

IF THE TANGENT PLANE IS HORIZONTAL, ITS NORMAL VECTOR IS PARALLEL TO $\hat{k}$.

LET $F(x,y,z) = xy + \frac{1}{x} + \frac{1}{y} - z$. OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$. $\vec{\nabla} F(x,y,z)$ IS NORMAL AT ANY POINT SATISFYING $F(x,y,z) = 0$.

$$\vec{\nabla} F(x,y,z) = \left(y - \frac{1}{x^2}\right)\hat{\imath} + \left(x - \frac{1}{y^2}\right)\hat{\jmath} - \hat{k}$$

$x = 1 \Rightarrow y = 1 \Rightarrow z = 3$

THIS IS PARALLEL TO $\hat{k}$ ONLY IF

$$\underbrace{y - \frac{1}{x^2} = 0 \quad \text{AND} \quad x - \frac{1}{y^2} = 0}_{x - x^4 = 0 \;\Rightarrow\; x=0,\; x=1}$$

$\boxed{\text{THE POINT IS } (1,1,3).}$

12. (10 points) Find and classify the critical points of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

$f_x(x,y) = -3x^2 + 4y$

$f_y(x,y) = 4x - 4y$

$-3x^2 + 4y = 0$

$\underbrace{4x - 4y = 0}_{x = y}$

$\Rightarrow -3x^2 + 4x = 0$

$-x(3x - 4) = 0$

$x = 0$ OR $x = \frac{4}{3}$

CRITICAL PTS ARE $(0,0)$ AND $\left(\frac{4}{3}, \frac{4}{3}\right)$.

$$D(x,y) = \begin{vmatrix} -6x & 4 \\ 4 & -4 \end{vmatrix}$$

$$= 24x - 16$$

$D(0,0) = -16$

$(0,0,1)$ IS A SADDLE PT.

$D\left(\frac{4}{3}, \frac{4}{3}\right) = 16$ & $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8$

$f\left(\frac{4}{3}, \frac{4}{3}\right) = \frac{59}{27}$ IS A REL. MAX.

13. (8 points) Find a set of parametric equations for the line normal to the graph of $\underbrace{y \ln(xz^2) = 2}$ at the point $(e, 2, 1)$.

THIS SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 2$, WHERE $F(x,y,z) = y \ln(xz^2)$.

$= y \ln x + 2y \ln z$

ALSO NOTICE THAT $F(e,2,1) = 2 \ln(e) = 2$.

$$\vec{\nabla} F(x,y,z) = \frac{y}{x}\hat{\imath} + \ln(xz^2)\hat{\jmath} + \frac{2y}{z}\hat{k}$$

$$\vec{n} = \vec{\nabla} F(e,2,1) = \frac{2}{e}\hat{\imath} + \hat{\jmath} + 4\hat{k}$$

NORMAL LINE:

$x = \frac{2}{e}t + e$

$y = t + 2$

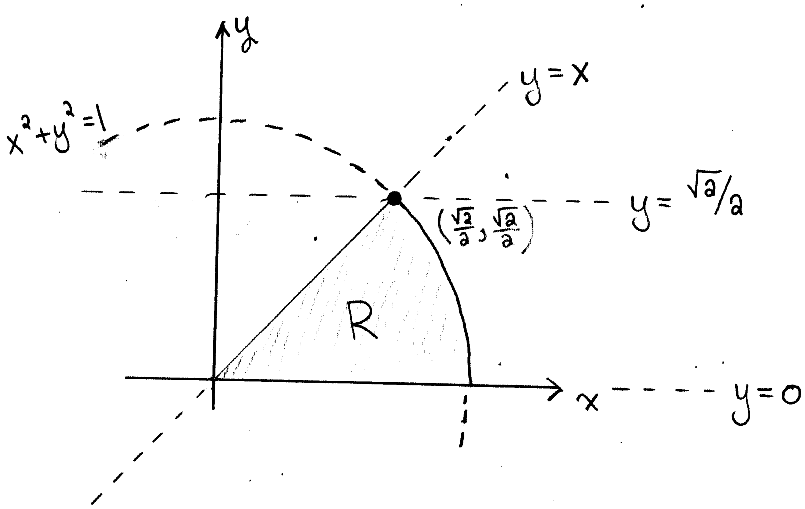
$z = 4t + 1$

$x = \sqrt{1-y^2}$

$\Rightarrow x^2 + y^2 = 1$

14. (10 points) Consider the iterated integral $\displaystyle\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} xy \, dx \, dy.$

(a) Sketch the region of integration (in detail).



(b) Evaluate the iterated integral.

$$\int_{y=0}^{y=\sqrt{2}/2} \frac{1}{2}x^2 y \Big|_{x=y}^{x=\sqrt{1-y^2}} dx = \int_0^{\sqrt{2}/2} \left(\frac{1}{2}(1-y^2)y - \frac{1}{2}y^3\right) dy$$

$$= \int_0^{\sqrt{2}/2} \left(\frac{1}{2}y - y^3\right) dy = \frac{1}{4}y^2 - \frac{1}{4}y^4 \Big|_0^{\sqrt{2}/2}$$

$$= \frac{2}{16} - \frac{1}{16} = \boxed{\frac{1}{16}}$$