# Math 233 - Final Exam A

December 9, 2022

Name key

Score ______

**Show all work to receive full credit**. Supply explanations where necessary. Place your final answer in the box provided. This test is due December 15. You must work individually.

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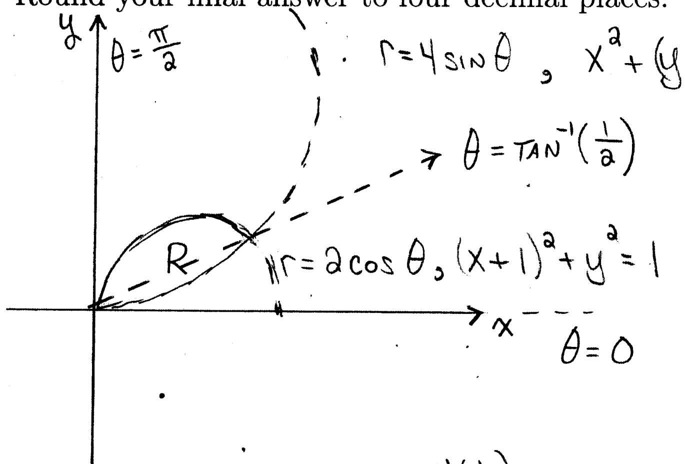
1. (5 points) Use a double integral to find the area of the 1st quadrant region inside both the circle $r = 2\cos\theta$ and the circle $r = 4\sin\theta$. After sketching the region and setting up the required iterated integral(s), you may use technology to evaluate the integral(s). Round your final answer to four decimal places.




$$4\sin\theta = 2\cos\theta$$

$$\Downarrow$$

$$\tan^{-1}\theta = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$

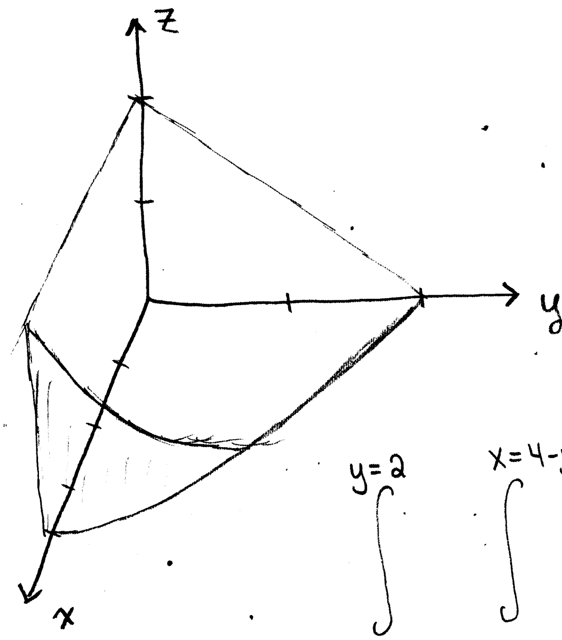
$$\text{AREA} = \int_{\theta=0}^{\theta=\tan^{-1}(\frac{1}{2})} \int_{r=0}^{r=4\sin\theta} r\,dr\,d\theta + \int_{\theta=\tan^{-1}(\frac{1}{2})}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} r\,dr\,d\theta$$

$$= \frac{1}{2}\pi + 3\tan^{-1}\left(\frac{1}{2}\right) - \frac{5}{2}\sin\left(2\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$\approx 0.961739$$

$$\boxed{\text{AREA} \approx 0.9617}$$

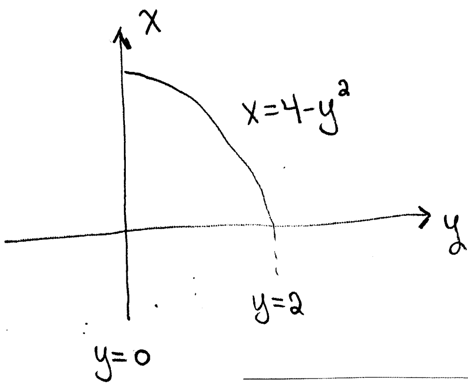
2. (5 points) A solid lies in the 1st octant bounded by the coordinates planes, the plane $y+z=2$, and the parabolic cylinder $x=4-y^2$. The density of the solid at the point $(x,y,z)$ is given by $\rho(x,y,z)=2+z+\sin x$. Set up the triple integral that gives the mass of the solid. Use technology to evaluate the integral. Round your final answer to four decimal places.




$$\text{Mass} = \iiint_S \rho(x,y,z)\,dV$$

$$\int_{y=0}^{y=2}\int_{x=0}^{x=4-y^2}\int_{z=0}^{z=2-y} (2+z+\sin x)\,dz\,dx\,dy$$

$$\approx 21.5763$$




My Sage code

```
var("x","y","z")
assume(y>0)
assume(y<2)
ANS=integrate(integrate(integrate(2+z+sin(x),z,0,2-y),x,0,4-y^2),y,0,2)
N(ANS)
```

$$\boxed{\int_0^2\int_0^{4-y^2}\int_0^{2-y}(2+z+\sin x)\,dz\,dx\,dy \approx 21.5763}$$

$\qquad\qquad\qquad\qquad M \qquad\quad N$

3. (10 points) Let $\vec{F}(x,y) = (y^3+1)\hat{\imath} + (3xy^2+1)\hat{\jmath}$.

   (a) Use our test to show that $\vec{F}$ is a conservative vector field. Then find a scalar potential function for $\vec{F}$. Write the potential function in the box below.

$$\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x} = 3y^2 \Rightarrow \vec{F} \text{ IS CONSERVATIVE.}$$

$$f_x(x,y) = y^3+1 \Rightarrow f(x,y) = xy^3 + x + g(y)$$

$$f_y(x,y) = 3xy^2+1 \Rightarrow f(x,y) = xy^3 + y + h(x)$$

$$\boxed{f(x,y) = xy^3 + x + y}$$

   (b) Evaluate $\int_C \vec{F}(x,y)\cdot d\vec{r}$ where $C$ is the semicircular path in the 1st quadrant from $(0,0)$ to $(2,0)$. (Hint: Use the fact that $\vec{F}$ is conservative.)

$$\int_C \vec{F}(x,y)\cdot d\vec{r} = f(2,0) - f(0,0)$$

$$= (0+2+0) - (0+0+0)$$

$$= 2$$

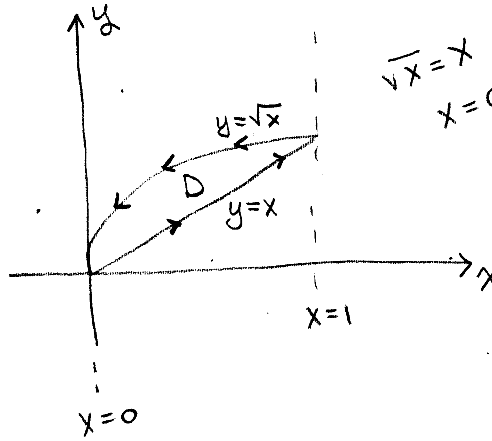
$$\boxed{2}$$

4. (5 points) Use Green's Theorem to evaluate

$$\int_C \overset{M}{\cos y}\, dx + (\overset{N}{xy - x\sin y})\, dy,$$

where $C$ is the positively-oriented boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$. Evaluate your integral by hand.

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$




$\sqrt{x} = x$

$x = 0, x = 1.$

$$\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} \left[y - \sin y - (-\sin y)\right] dy\, dx$$

$$= \int_0^1 \int_x^{\sqrt{x}} y\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 (x - x^2)\, dx = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{6}\right) = \frac{1}{12}$$

$$\boxed{\frac{1}{12}}$$

5. (5 points) Find the critical point(s) and relative extreme value(s) of

$$f(x, y) = xy + 1/x + 8/y.$$

$$f_x(x,y) = y - \frac{1}{x^2}$$

$$f_y(x,y) = x - \frac{8}{y^2}$$

$$f_x = 0 \Rightarrow y = \frac{1}{x^2}.$$

$$f_y = 0 \Rightarrow x - 8x^4 = 0$$

$$x(1-8x^3) = 0$$

$$\cancel{x=0} \quad \text{OR} \quad x = \frac{1}{2}$$

$$y = 4$$

ONLY ONE CRIT. PT.

$\left(\frac{1}{2}, 4\right)$

$$D = \begin{vmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{16}{y^3} \end{vmatrix} = \frac{32}{x^3y^3} - 1$$

$$D\left(\tfrac{1}{2}, 4\right) = 4 - 1 = 3 > 0$$

AND $f_{xx}\left(\tfrac{1}{2}, 4\right) = 16 > 0$

$\Longrightarrow$ $f\left(\frac{1}{2}, 4\right) = 6$ IS A REL MIN.

| | |
|---|---|
| CRIT PT IS $\left(\frac{1}{2}, 4\right)$. | $f\left(\frac{1}{2}, 4\right) = 6$ IS A RELATIVE MINIMUM. |