Math 233 - Final Exam B Name key
December 15, 2022 Score ____________

Show all work to receive full credit. Supply explanations where necessary. Place your final answer in the box provided.

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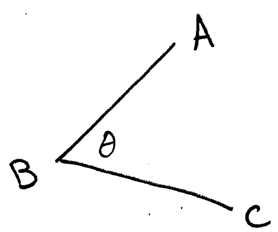
1. (5 points) The points A, B, and C are the vertices of a triangle in space. Find the measure of the angle at B. Write your answer in degrees, rounded to the nearest hundredth.

$$A\,(0,-3,7), \qquad B\,(1,-2,3), \qquad C\,(5,1,-2)$$



$\overrightarrow{BA} = -\hat{\imath} - \hat{\jmath} + 4\hat{k}$

$\overrightarrow{BC} = 4\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\|\overrightarrow{BA}\|\,\|\overrightarrow{BC}\|} = \frac{-27}{\sqrt{18}\sqrt{50}}$$

$\theta \approx 154.16^\circ$

$$\boxed{\theta \approx 154.16^\circ}$$

2. (5 points) A parallelepiped is determined by the vectors $\vec{u} = 3\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$, $\vec{v} = 2\hat{\imath}+\hat{\jmath}-5\hat{k}$, and $\vec{w} = -3\hat{\imath}+2\hat{\jmath}-3\hat{k}$. Find the area of the face that is the parallelogram determined by $\vec{u}$ and $\vec{w}$.

$\text{Area} = \|\vec{u}\times\vec{w}\|$

$$\vec{u}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -4 & 2 \\ -3 & 2 & -3 \end{vmatrix}$$

$$= \hat{\imath}(8) - \hat{\jmath}(-3) + \hat{k}(-6)$$

$$\|\vec{u}\times\vec{w}\| = \sqrt{8^2 + 3^2 + (-6)^2} = \sqrt{109}$$

$$\boxed{\sqrt{109} \approx 10.44}$$

3. (5 points) Find symmetric equations for the line through $P\,(17, 9, -11)$ and $Q\,(12, -1, -10)$.

$\overrightarrow{QP} = 5\hat{\imath} + 10\hat{\jmath} - \hat{k}$

Using Q & $\overrightarrow{QP}$...

$$\frac{x-12}{5} = \frac{y+1}{10} = \frac{z+10}{-1}$$

$$\boxed{\frac{x-12}{5} = \frac{y+1}{10} = \frac{z+10}{-1}}$$

4. (5 points) An object is moved along the graph of

$$\vec{r}(t) = t^2\,\hat{\imath} + \cos(\pi t)\,\hat{\jmath} + \sqrt{t}\,\hat{k}$$

from the point $(0, 1, 0)$ to the point $(16, 1, 2)$. Set up the definite integral that gives the length of the path. Use you calculator to approximate the value of the integral.

$(0,1,0) \Rightarrow t=0$

$(16,1,2) \Rightarrow t=4$

$\vec{r}\,'(t) = 2t\,\hat{\imath} - \pi \sin(\pi t)\,\hat{\jmath} + \frac{1}{2}t^{-1/2}\,\hat{k}$

$$\text{Arc length} = \int_0^4 \sqrt{4t^2 + \pi^2 \sin^2(\pi t) + \frac{1}{4t}}\; dt$$

← Integral is actually improper.

$$\approx 19.31$$

$$\boxed{\int_0^4 \sqrt{4t^2 + \pi^2 \sin^2(\pi t) + \frac{1}{4t}}\; dt \approx 19.31}$$

5. (5 points) Ralphie shoots his Red Ryder BB gun from 4 ft above the ground across a level field while holding the barrel of his BB gun at an angle of 5° above horizontal. Assuming the BB's muzzle velocity is 350 ft/s, how far downrange will the BB travel before hitting the ground?

$$\vec{r}(t) = 350\cos(5°)t\,\hat{i} + (-16t^2 + 350\sin(5°)t + 4)\,\hat{j}$$

$$-16t^2 + 350\sin(5°)t + 4 = 0$$

$$\Rightarrow t = \frac{-350\sin(5°) - \sqrt{(350\sin 5°)^2 - 4(-16)(4)}}{-32}$$

$$t \approx 2.0297$$

$$350\cos(5°)(2.0297)$$

$$\approx 707.69 \text{ FT}$$

$$\boxed{707.69 \text{ FT}}$$

6. (5 points) Find the limit or show that it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$ %

MORE WORK

CONVERT TO POLAR...

$$\lim_{r\to 0} \frac{r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r\sin^3\theta = \sin^3\theta \lim_{r\to 0} r$$

$$= 0$$

$$\boxed{0}$$

% More work

7. (5 points) Find the limit or show that it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Along $x=0$ ... $\displaystyle\lim_{y\to 0} \frac{0}{y^2} = 0$

Along $x=y$ ... $\displaystyle\lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

$\boxed{\text{Limit DNE by two-path test.}}$

8. (5 points) The period, T, of a pendulum of length L is given by

$$T = \frac{2\pi\sqrt{L}}{\sqrt{g}},$$

where g is the acceleration due to gravity. A pendulum is moved from a location where $g = 32.09\,\text{ft/s}^2$ to a location where $g = 32.23\,\text{ft/s}^2$. There was also a temperature change that resulted in a change in the length of the pendulum from 2.5 ft to 2.48 ft. Use differentials to approximate the corresponding change in the pendulum's period.

$$\Delta T \approx \frac{\pi}{\sqrt{Lg}}\Delta L + \frac{-\pi\sqrt{L}}{(g)^{3/2}}\Delta g$$

$g = 32.09$, $\Delta g = 0.14$, $L = 2.5$, $\Delta L = -0.02$

$$\Delta T \approx \frac{\pi(-0.02)}{\sqrt{(2.5)(32.09)}} - \frac{\pi\sqrt{2.5}\,(0.14)}{(32.09)^{3/2}} \approx -0.01084$$

$\boxed{\Delta T \approx -0.01084 \text{ seconds}}$

9. (5 points) Find an equation of the plane tangent to the surface $ze^{x^2-y^2} - 3 = 0$ at the point $P\,(2,2,3)$.

LET $F(x,y,z) = z e^{x^2-y^2}$

OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 3$.

$$\vec{\nabla} F(x,y,z) = 2xze^{x^2-y^2}\,\hat{\imath} - 2yze^{x^2-y^2}\,\hat{\jmath} + e^{x^2-y^2}\,\hat{k}$$

$$\vec{n} = \vec{\nabla} F(2,2,3) = 12\hat{\imath} - 12\hat{\jmath} + \hat{k}$$

$$12x - 12y + z = 12(2) - 12(2) + 3$$

$$\boxed{12x - 12y + z = 3}$$

10. (5 points) Find the directional derivative of

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

at the point $(5,-5,5)$ in the $\underbrace{\text{direction the function decreases most rapidly.}}_{-\vec{\nabla} F(5,-5,5)}$

$$\vec{\nabla} F(x,y,z) = \left(\frac{1}{y} - \frac{z}{x^2}\right)\hat{\imath} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\hat{\jmath} + \left(\frac{1}{x} - \frac{y}{z^2}\right)\hat{k}$$

$$\vec{\nabla} F(5,-5,5) = -\frac{2}{5}\hat{\imath} + 0\hat{\jmath} + \frac{2}{5}\hat{k}$$

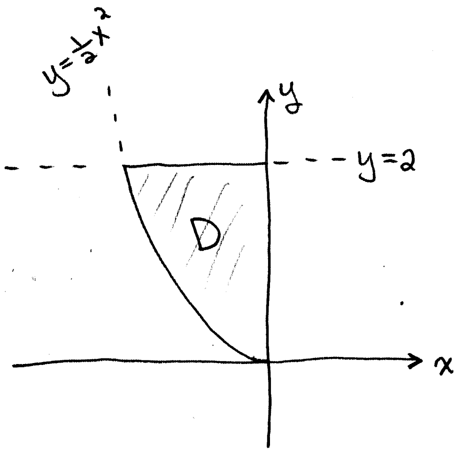
DIRECTIONAL DERIV $= \dfrac{\vec{\nabla} F \cdot \vec{\nabla} F}{\|\vec{\nabla} F\|} = -\|\vec{\nabla} F\| = -\sqrt{\dfrac{4}{25} + \dfrac{4}{25}}$

$$= -\frac{\sqrt{8}}{5} = \frac{-2\sqrt{2}}{5}$$

$$\boxed{-\frac{2\sqrt{2}}{5} \approx -0.56569}$$

11. (5 points) Let D be the region in the second quadrant bounded by graphs of $y = \frac{1}{2}x^2$, $y = 2$, and $x = 0$. Write the double integral as an iterated integral and evaluate.

$$\iint_D x^5 \sin(y^4)\, dA$$



$$\int_{y=0}^{y=2}\int_{x=-\sqrt{2y}}^{x=0} x^5 \sin(y^4)\, dx\, dy = \int_0^2 \frac{1}{6}x^6\Big|_{-\sqrt{2y}}^{0} \sin(y^4)\, dy$$

$$= \int_0^2 -\frac{8}{6}y^3 \sin(y^4)\, dy$$

$u = y^4$

$du = 4y^3\, dy$

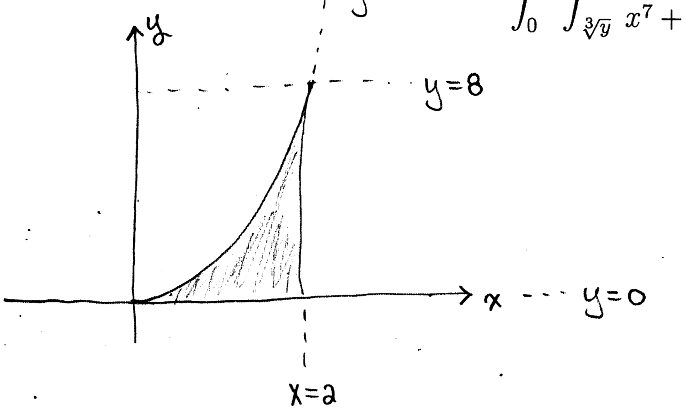
$$= \int_0^{16} -\frac{1}{3}\sin u\, du = \frac{1}{3}\cos u \Big|_0^{16}$$

$$\boxed{\frac{1}{3}\cos(16) - \frac{1}{3} \approx -0.65255}$$

12. (5 points) Sketch the region of integration. Then reverse the order of integration and evaluate.

$y = x^3,\ x = \sqrt[3]{y}$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y}{x^7+1}\, dx\, dy$$



$$\int_{x=0}^{x=2}\int_{y=0}^{y=x^3} \frac{y}{x^7+1}\, dy\, dx = \int_0^2 \frac{1}{2}\frac{(x^3)^2}{x^7+1}\, dx$$

$u = x^7 + 1$

$du = 7x^6\, dx$

$\frac{1}{7}du = x^6\, dx$

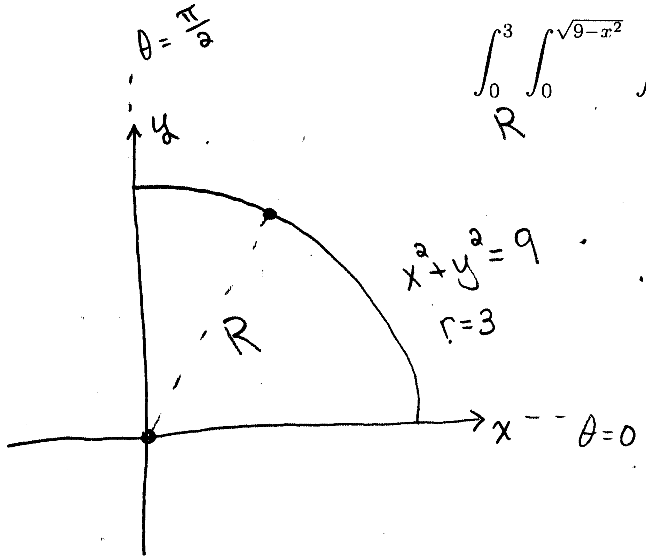
$$\frac{1}{14}\int_1^{129} \frac{1}{u}\, du$$

$$\boxed{\frac{1}{14}\ln(129) \approx 0.347129}$$

$$y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9$$

13. (5 points) Convert the iterated integral to an equivalent integral in cylindrical coordinates. DO NOT EVALUATE.

$$\underbrace{\int_0^3 \int_0^{\sqrt{9-x^2}}}_{R} \int_{-\sqrt{2x^2+2y^2}}^{6+x^2+y^2} 15z\,dz\,dy\,dx = \iint_R \int_{z=-\sqrt{2r^2}}^{z=6+r^2} 15z\,dz\,r\,dA$$



$x^2+y^2=9$

$r=3$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=3} \int_{z=-\sqrt{2}r}^{z=6+r^2} 15rz\,dz\,dr\,d\theta$$

$$\boxed{\int_0^{\pi/2} \int_0^3 \int_{-\sqrt{2}r}^{6+r^2} 15rz\,dz\,dr\,d\theta}$$

14. (5 points) Evaluate

$$\int_C y\,dx + x^2\,dy,$$

where C is the parabolic arc along $y = 4x - x^2$ from $(4,0)$ to $(1,3)$.

$$dy = (4-2x)\,dx$$

$$\int_{x=4}^{x=1} (4x-x^2)\,dx + x^2(4-2x)\,dx = \int_4^1 (4x + 3x^2 - 2x^3)\,dx$$

$$= 2x^2 + x^3 - \frac{1}{2}x^4 \Big|_4^1 = \left(2+1-\frac{1}{2}\right) - (32+64-128) = \frac{69}{2}$$

$$\boxed{\frac{69}{2} = 34.5}$$