Math 233 - Test 1
September 14, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) In this problem, the force vectors $\vec{F_1}$ and $\vec{F_2}$ are 2D vectors in the xy-plane.

   (a) The force $\vec{F_1}$ has magnitude 50 and makes a 120° angle with the positive x-axis. Find the component form of $\vec{F_1}$.

   $$\vec{F_1} = 50\cos 120^\circ\,\hat{\imath} + 50\sin 120^\circ\,\hat{\jmath}$$
   $$= \boxed{-25\,\hat{\imath} + 25\sqrt{3}\,\hat{\jmath}}$$

   (b) The force $\vec{F_2}$ has component form $\vec{F_2} = 30\hat{\imath} - 30\sqrt{3}\,\hat{\jmath}$. What angle does $\vec{F_2}$ make with the positive x-axis?

   $$\|\vec{F_2}\| = \sqrt{(30)^2 + (-30\sqrt{3})^2} = \sqrt{3600} = 60$$
   $$\cos\theta = \frac{30}{60} \text{ AND } \theta \text{ IN QUAD IV} \Rightarrow \boxed{\theta = -60^\circ}$$

   (c) Refer to parts (a) and (b). Compute the resultant vector $\vec{F} = \vec{F_1} + \vec{F_2}$.

   $$\vec{F_1} + \vec{F_2} = (-25+30)\,\hat{\imath} + (25\sqrt{3} - 30\sqrt{3})\,\hat{\jmath}$$
   $$= \boxed{5\hat{\imath} - 5\sqrt{3}\,\hat{\jmath}}$$

   (d) Refer to part (c). What angle does $\vec{F}$ make with the positive x-axis?

   NOTICE THAT $\vec{F} = \frac{1}{6}\vec{F_2}$.
   $$\Rightarrow \boxed{\theta = -60^\circ}$$

2. (4 points) Explain how the right-hand rule gives the orientation of the coordinate axes in a 3-dimensional rectangular coordinate system.

USING YOUR RIGHT HAND, POINT YOUR FINGERS IN THE DIRECTION OF THE POSITIVE X-AXIS. THEN CURL YOUR FINGERS IN THE DIRECTION OF THE POSITIVE Y-AXIS, AND YOUR THUMB WILL POINT IN THE DIRECTION OF THE POSITIVE Z-AXIS.

Using $\vec{u}\cdot\vec{w} = \|\vec{u}\|\,\|\vec{w}\|\cos\theta$

3. (6 points) Suppose that θ is the angle between the two nonzero vectors $\vec{u}$ and $\vec{w}$. What can you say about $\vec{u}\cdot\vec{w}$ in each of these cases?

   (a) θ is a right angle. $\boxed{\vec{u}\cdot\vec{w} = 0}$

   (b) θ is an acute angle. $\boxed{\vec{u}\cdot\vec{w} > 0}$

   (c) θ is an obtuse angle. $\boxed{\vec{u}\cdot\vec{w} < 0}$

4. (6 points) Find a vector of magnitude 6 that has the direction from $P\,(-2,4,-3)$ to $Q\,(-5,3,4)$.

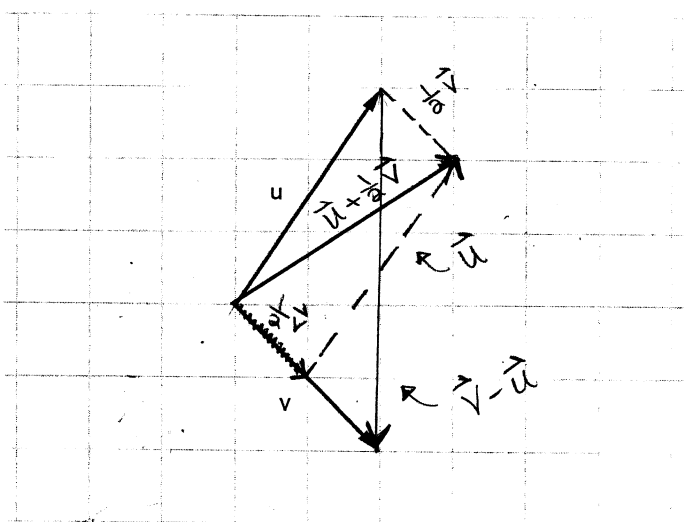
$$\vec{PQ} = (-5+2)\hat{i} + (3-4)\hat{j} + (4+3)\hat{k}$$

$$= -3\hat{i} - \hat{j} + 7\hat{k}$$

$$\|\vec{PQ}\| = \sqrt{9+1+49} = \sqrt{59}$$

$$\boxed{\frac{6\,\vec{PQ}}{\|\vec{PQ}\|} = \frac{-18}{\sqrt{59}}\hat{i} - \frac{6}{\sqrt{59}}\hat{j} + \frac{42}{\sqrt{59}}\hat{k}}$$

5. (6 points) The figure below shows the vectors $\vec{u}$ and $\vec{v}$. Sketch and label the vectors $\vec{u} + \frac{1}{2}\vec{v}$ and $\vec{v} - \vec{u}$, and illustrate how your vectors follow from the parallelogram law.



$\vec{u} + \frac{1}{2}\vec{v}$ IS A DIAGONAL OF THE PARALLELOGRAM FORMED BY $\vec{u}$ & $\frac{1}{2}\vec{v}$.

$\vec{v} - \vec{u}$ IS A DIAGONAL OF THE PARALLELOGRAM FORMED BY $\vec{u}$ AND $\vec{v}$.

6. (6 points) Let $\vec{x} = -9\hat{\imath} - 5\hat{\jmath} + 3\hat{k}$.

(a) Find a vector, different from $\vec{x}$, that is parallel to $\vec{x}$. Give a one-sentence explanation for how you know.

$-2\vec{x} = \boxed{18\hat{i} + 10\hat{j} - 6\hat{k}}$

TWO VECTORS ARE PARALLEL IFF ONE IS A NONZERO SCALAR MULTIPLE OF THE OTHER.

(b) Find a nonzero vector that is orthogonal to $\vec{x}$. Give a one-sentence explanation for how you know.

$\vec{y} = \boxed{\hat{i} + 3\hat{k}} \Rightarrow \vec{x} \cdot \vec{y} = -9 + 0 + 9 = 0$

DOT PROD ZERO MEANS THE VECTORS ARE ORTHOGONAL.

7. (10 points) For this problem, you will need to use that the distance from a point Q to the line passing through P and parallel to $\vec{v}$ is given by

$$D = \frac{\|\vec{PQ} \times \vec{v}\|}{\|\vec{v}\|}.$$

ANSWERS VARY FOR THIS PROBLEM.

(a) First choose any point on the line described by the parametric equations below. Let your point be Q. (There are infinitely many choices for Q.)

$$x = 3t - 4, \quad y = -5t, \quad z = t + 5.$$

$\boxed{Q(-4, 0, 5)}$ (CORRESPONDS WITH $t = 0$)

(b) Now consider the line ℓ with symmetric equations

$$\frac{x+6}{2} = y - 3 = \frac{z-1}{-3}.$$

Find a point P on ℓ and a vector $\vec{v}$ parallel to ℓ.

$P(-6, 3, 1)$ ← READ FROM NUMERATORS

$\vec{v} = 2\hat{i} + \hat{j} - 3\hat{k}$ ← READ FROM DENOMINATORS

(c) Compute the distance from Q to the line ℓ.

$\vec{PQ} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 2 & 1 & -3 \end{vmatrix}$$

$= 5\hat{i} + 14\hat{j} + 8\hat{k}$

$\|\vec{PQ} \times \vec{v}\| = \sqrt{25 + 196 + 64} = \sqrt{285}$

$\|\vec{v}\| = \sqrt{4 + 1 + 9} = \sqrt{14}$

$\boxed{D = \sqrt{\frac{285}{14}} \approx 4.51}$

8. (8 points) Find the angle between the planes described by the equations below. Write your final answer in degrees rounded to the nearest hundredth.

$$2x - y + 2z = 7 \qquad\qquad -5x + 3z = 12$$

$\vec{n}_1 = 2\hat{i} - \hat{j} + 2\hat{k}$ $\qquad \|\vec{n}_1\| = \sqrt{4+1+4} = 3$

$\vec{n}_2 = -5\hat{i} + 0\hat{j} + 3\hat{k}$ $\qquad \|\vec{n}_2\| = \sqrt{25+9} = \sqrt{34}$

$\vec{n}_1 \cdot \vec{n}_2 = -10 + 6 = -4$ $\qquad \cos\theta = \dfrac{4}{3\sqrt{34}} \Rightarrow \boxed{\theta \approx 76.78^\circ}$

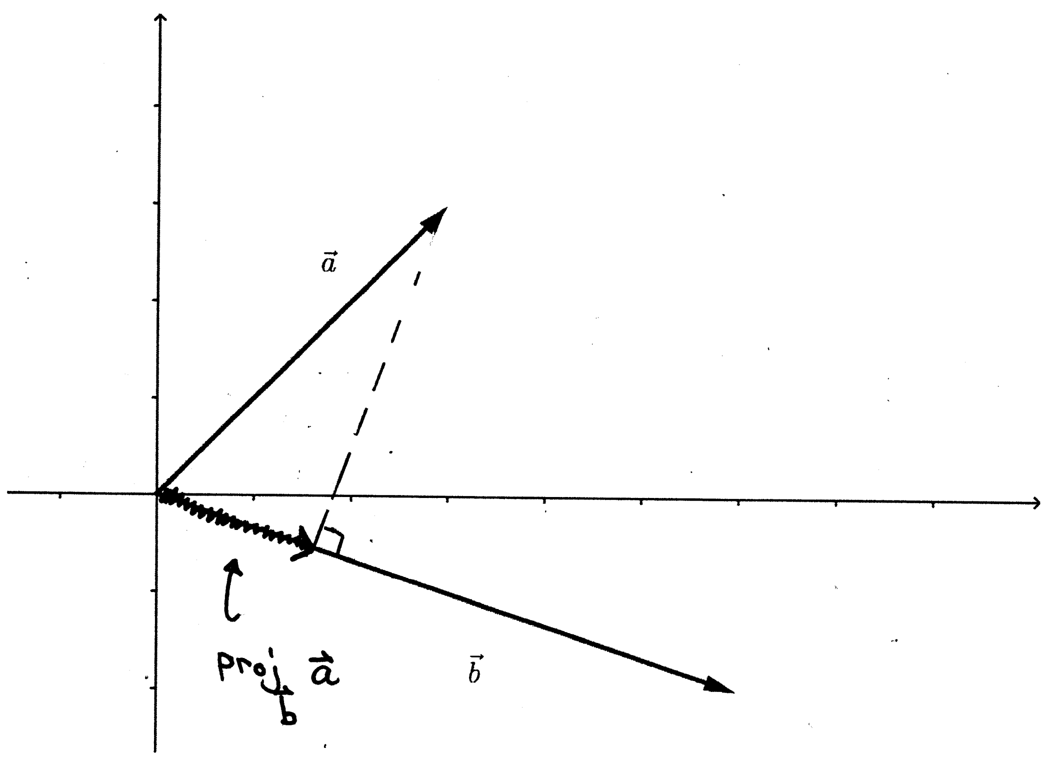
9. (4 points) Find the projection of $\vec{w} = \hat{\imath} + 4\hat{\jmath} - 3\hat{k}$ onto $\vec{u} = 7\hat{\imath} + 4\hat{k}$.

$\text{proj}_{\vec{u}}\,\vec{w} = \dfrac{\vec{w}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \dfrac{-5}{65}\,\vec{u} = \boxed{-\dfrac{1}{13}(7\hat{i} + 4\hat{k}) = -\dfrac{7}{13}\hat{i} - \dfrac{4}{13}\hat{k}}$

$\vec{w}\cdot\vec{u} = 7 + 0 - 12 = -5$

$\vec{u}\cdot\vec{u} = 49 + 0 + 16 = 65$

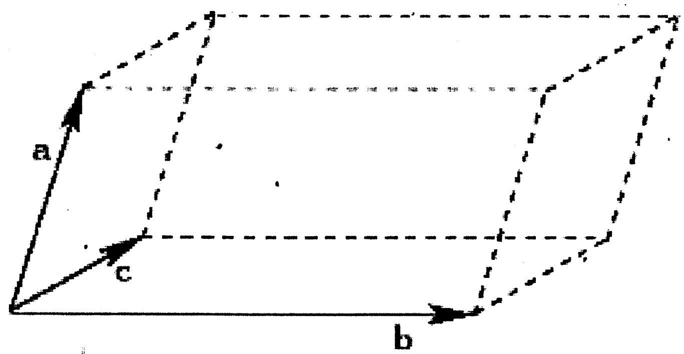
10. (4 points) The figure below shows the vectors $\vec{a}$ and $\vec{b}$. Sketch $\text{proj}_{\vec{b}}\,\vec{a}$.



11. (8 points) A crystal structure has the form of a parallelepiped determined by the vectors $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\vec{b} = 3\hat{\jmath} + 5\hat{k}$, and $\vec{c} = -4\hat{\imath} + 2\hat{\jmath} + \hat{k}$, where distances are measured in micrometers. Find the volume of the parallelepiped.

VOLUME

$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$



$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ -4 & 2 & 1 \end{vmatrix} = 1(3-10) - 2(0+20) + 1(0+12)$$

$$= -7 - 40 + 12 = -35$$

$\boxed{\text{VOLUME} = 35 \ \mu m^3}$

12. (6 points) Let $\vec{r}(t) = \frac{\sin t}{t}\hat{\imath} + \ln(t+1)\hat{\jmath} + e^{2t}\hat{k}$.

(a) Determine the domain of $\vec{r}$.

$\boxed{(-1,0) \cup (0,\infty)} = \{t : t > 1 \text{ AND } t \neq 0\}$

(b) Compute $\lim_{t\to 0} \vec{r}(t)$.

$$\lim_{t\to 0} \frac{\sin t}{t}\hat{\imath} + \lim_{t\to 0} \ln(t+1)\hat{\jmath} + \lim_{t\to 0} e^{2t}\hat{k}$$

$$= 1\hat{\imath} + 0\hat{\jmath} + 1\hat{k} = \boxed{\hat{\imath} + \hat{k}}$$

13. (4 points) Explain how to find a vector that is orthogonal to each vector in a pair of non-parallel vectors.

THE CROSS PRODUCT OF $\vec{u}$ & $\vec{v}$ IS ORTHOG. TO BOTH.

$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ AND $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$

14. (8 points) Let $\vec{r}(t) = 3\sin t\,\hat{\imath} - 3\cos t\,\hat{\jmath} + 4\,\hat{k}$.

(a) Compute $\|\vec{r}(t)\|$.

$$\|\vec{r}(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = \boxed{5}$$

(b) Determine the derivative $\vec{r}\,'(t)$.

$$\boxed{\vec{r}\,'(t) = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath}}$$

(c) Compute $\vec{r}(t) \cdot \vec{r}\,'(t)$.

$$\vec{r} \cdot \vec{r}\,' = 9\sin t\cos t - 9\cos t\sin t + 0$$
$$= \boxed{0}$$

$\vec{r}$ is orthog. to $\vec{r}\,'$

(d) Compute $\vec{r}(t) \times \vec{r}\,'(t)$.

$$\vec{r} \times \vec{r}\,' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3\sin t & -3\cos t & 4 \\ 3\cos t & 3\sin t & 0 \end{vmatrix} = \hat{\imath}(-12\sin t) - \hat{\jmath}(-12\cos t) + \hat{k}(9\sin^2 t + 9\cos^2 t)$$
$$= -12\sin t\,\hat{\imath} + 12\cos t\,\hat{\jmath} + 9\,\hat{k}$$

15. (10 points) Find an equation of the plane passing through the points $R(1, -2, 4)$, $S(0, 3, -5)$, and $T(8, 2, -3)$.

$$\overrightarrow{RS} = -\hat{\imath} + 5\hat{\jmath} - 9\hat{k}$$
$$\overrightarrow{RT} = 7\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$$

$$\vec{n} = \overrightarrow{RS} \times \overrightarrow{RT} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 5 & -9 \\ 7 & 4 & -7 \end{vmatrix}$$

$$= \hat{\imath}(-35+36) - \hat{\jmath}(7+63) + \hat{k}(-4-35)$$
$$= \hat{\imath} - 70\hat{\jmath} - 39\hat{k}$$

Plane is

$$x - 70y - 39z = 1 - 70(-2) - 39(4)$$
$$= -15$$

$$x - 70y - 39z = -15$$