Math 233 - Test 2
October 12, 2023

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (4 points) Describe, in detail, the graph of the vector-valued function
$$\vec{r}(t) = (1+t)\hat{\imath} + (2+5t)\hat{\jmath} + (-1+6t)\hat{k}.$$

$x = 1+t$
$y = 2+5t$
$z = -1+6t$

$\Rightarrow$ THE GRAPH IS THE LINE IN SPACE PASSING THROUGH $(1,2,-1)$ AND PARALLEL TO THE VECTOR $\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$.

2. (8 points) Consider the vector-valued function $\vec{r}(t) = (t+2)\hat{\imath} + (t^2-3)\hat{\jmath}$.

   (a) Write a corresponding set of parametric equations whose graph is the same as that of $\vec{r}$.

   $x = t+2$
   $y = t^2-3$

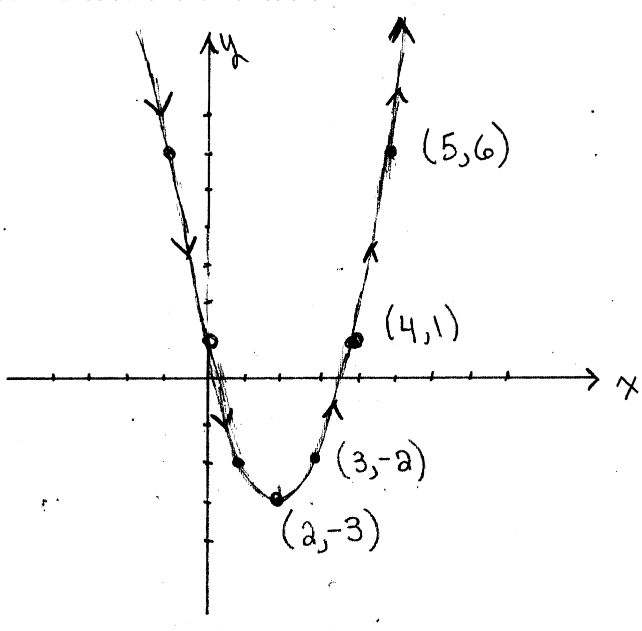
   (b) Eliminate the parameter t from your parametric equations above to find an equation for the graph in terms of the variables x and y.

   $t = x-2 \Rightarrow \boxed{y = (x-2)^2 - 3}$

   (c) Sketch the graph of $\vec{r}$ and use arrows to indicate the orientation.

THE GRAPH IS THE PARABOLA $y=x^2$ SHIFTED 2 UNITS RIGHT AND 3 UNITS DOWN.

SINCE $t = x-2$, THE ORIENTATION IS IN THE DIRECTION OF INCREASING X.



$\vec{r}(0) = 0\hat{i} + \hat{j} - \hat{k}$

3. (10 points) An object is launched from the point $P(0, 1, -1)$ with initial velocity vector $\vec{v}(0) = \hat{\imath} - 2\hat{\jmath} + \hat{k}$. The object undergoes a constant acceleration of $\vec{a}(t) = \hat{\jmath} + 2\hat{k}$. Find the object's location at $t = 4$.

$$\vec{a}(t) = \hat{j} + 2\hat{k} \Rightarrow \vec{v}(t) = c_1\hat{i} + (t + c_2)\hat{j} + (2t + c_3)\hat{k}$$

$$\vec{v}(0) = \hat{i} - 2\hat{j} + \hat{k} \Rightarrow c_1 = 1,\ c_2 = -2,\ c_3 = 1$$

$$\vec{v}(t) = \hat{i} + (t-2)\hat{j} + (2t+1)\hat{k} \Rightarrow \vec{r}(t) = (t + d_1)\hat{i} + \left(\tfrac{1}{2}t^2 - 2t + d_2\right)\hat{j} + (t^2 + t + d_3)\hat{k}$$

$$\vec{r}(0) = 0\hat{i} + \hat{j} - \hat{k} \Rightarrow d_1 = 0,\ d_2 = 1,\ d_3 = -1$$

$$\vec{r}(t) = t\hat{i} + \left(\tfrac{1}{2}t^2 - 2t + 1\right)\hat{j} + (t^2 + t - 1)\hat{k}$$

$$\boxed{\vec{r}(4) = 4\hat{i} + \hat{j} + 19\hat{k}} \quad \text{OR} \quad \boxed{(4, 1, 19)}$$

4. (10 points) Set up the definite integral that gives the length of the graph of

$$\vec{r}(t) = (3t^2 + 1)\,\hat{\imath} + (4t^2 - 1)\,\hat{\jmath} + 4t^3\,\hat{k}$$

from $t = 0$ to $t = 2$. Evaluate your integral by hand. (If you've done everything correctly, your integral should require a simple u-substitution.)

$$\vec{r}'(t) = 6t\hat{i} + 8t\hat{j} + 12t^2\hat{k}, \quad t \geq 0$$

$$\|\vec{r}'(t)\| = \sqrt{36t^2 + 64t^2 + 144t^4} = \sqrt{100t^2 + 144t^4} = 2t\sqrt{25 + 36t^2}$$

$$S = \int_0^2 2t\sqrt{25 + 36t^2}\,dt = \frac{1}{36}\int_{25}^{169} u^{1/2}\,du = \frac{1}{54}u^{3/2}\Big|_{25}^{169}$$

$u = 25 + 36t^2$

$du = 72t\,dt$

$\frac{1}{36}du = 2t\,dt$

$$= \frac{1}{54}\left[13^3 - 5^3\right]$$

$$= \boxed{\frac{2072}{54}} \approx 38.37$$

5. (10 points) Let $\vec{r}(t) = (t-3)\,\hat{\imath} + (2t-4)\,\hat{\jmath} + 2t\,\hat{k}$. Reparameterize $\vec{r}$ in terms of the arc-length parameter starting from $t = 3$.

$$\vec{r}\,'(t) = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{1+4+4} = 3$$

$$s = \int_3^t 3\,du = 3u\Big|_3^t = 3t-9$$

$$s = 3t-9 \Rightarrow t = \frac{s+9}{3} = \frac{s}{3}+3$$

$$\boxed{\vec{R}(s) = \frac{s}{3}\hat{\imath} + \left(\frac{2s}{3}+2\right)\hat{\jmath} + \left(\frac{2s}{3}+6\right)\hat{k}, \quad s \geq 0}$$

6. (8 points) Show that the curvature at any point on the graph of $\vec{r}(t) = -2\sin(2t)\,\hat{\imath} - 2\cos(2t)\,\hat{\jmath}$ is 1/2.

$$\vec{r}\,'(t) = -4\cos(2t)\,\hat{\imath} + 4\sin(2t)\,\hat{\jmath}$$

$$\|\vec{r}\,'(t)\| = \sqrt{16\cos^2(2t) + 16\sin^2(2t)} = \sqrt{16} = 4$$

$$\hat{T}(t) = -\cos(2t)\,\hat{\imath} + \sin(2t)\,\hat{\jmath}$$

$$\hat{T}'(t) = 2\sin(2t)\,\hat{\imath} + 2\cos(2t)\,\hat{\jmath}$$

$$\|\hat{T}'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{4} = 2$$

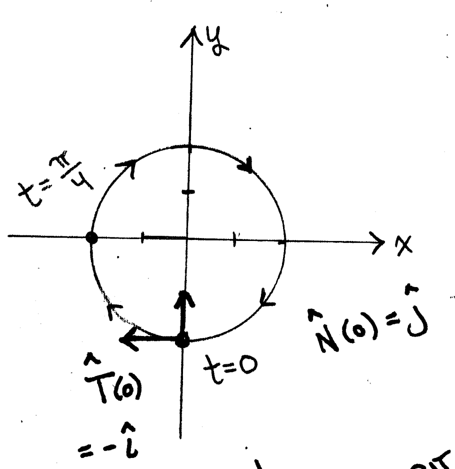
$$K(t) = \frac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{2}{4}$$

$$\boxed{K(t) = \frac{1}{2}}$$

7. (8 points) Sketch the graph of the vector-valued function $\vec{r}(t) = -2\sin(2t)\,\hat{\imath} - 2\cos(2t)\,\hat{\jmath}$. Then, on your graph, sketch the vectors $\hat{T}(0)$ and $\hat{N}(0)$.

GRAPH OF $\vec{r}(t)$ IS THE CIRCLE OF RADIUS 2 CENTERED AT (0,0)

t	$\vec{r}(t)$
0	$-2\hat{\jmath}$
$\frac{\pi}{4}$	$-2\hat{\imath}$



WHOOPS! MADE $\hat{T}(0)$ A BIT TOO LONG.

$$\vec{r}(t) = 500(\cos 60^\circ)t\,\hat{i} + \left(-4.9t^2 + 500(\sin 60^\circ)t\right)\hat{j}$$

8. (15 points) A projectile is fired into the air from ground level with an initial speed of 500 m/sec at an angle of 60° with the horizontal. (Use $g = 9.8\,\text{m/sec.}$)

(a) Find the maximum height of the projectile.

$$\vec{r}_y(t) = -4.9t^2 + 250\sqrt{3}\,t$$

$$\vec{r}_y\,'(t) = -9.8t + 250\sqrt{3} = 0.$$

$$t = \frac{250\sqrt{3}}{9.8} \approx 44.185 \text{ sec}$$

$$\vec{r}_y(44.185) \approx \boxed{9566.33 \text{ m}}$$

(b) What is the range of the projectile?

$$\vec{r}_y(t) = 0 \Rightarrow t\left(-4.9t + 250\sqrt{3}\right) = 0$$

$$t = 0 \text{ or } t = \frac{250\sqrt{3}}{4.9} \approx 88.3699 \text{ sec}$$

$$\vec{r}_x(88.3699) = (250)(88.3699) \approx \boxed{22{,}092.5 \text{ m}}$$

(c) Show that the speed of the projectile when it hits the ground is 500 m/sec.

$$\vec{r}\,'(t) = 250\hat{i} + \left(-9.8t + 250\sqrt{3}\right)\hat{j}$$

$$\vec{r}\,'\left(\frac{250\sqrt{3}}{4.9}\right) = 250\hat{i} + \left(-250\sqrt{3}\right)\hat{j}$$

$$\left\|\vec{r}\,'\left(\frac{250\sqrt{3}}{4.9}\right)\right\| = \sqrt{250^2 + 250^2\cdot 3} = \sqrt{250000} = 500\ \checkmark$$

9. (12 points) Each of these equations defines a surface in 3-space. Describe each surface.

(a) $4y = x^2 + 8z^2$ Fix y: Ellipses. Fix x: Parabolas. Fix z: Parabolas.

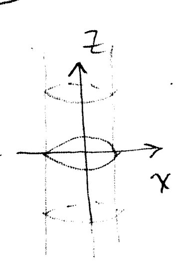
$\boxed{\text{Elliptical Paraboloid opening up } y\text{-axis, vertex at } (0,0,0)}$

(b) $z = 9x - 7y + 13$

$9x - 7y - z = -13$ $\boxed{\text{Plane with normal vector } 9\hat{i} - 7\hat{j} - \hat{k}}$

(c) $x^2 + y^2 = 4$

z is arbitrary. $\boxed{\text{Circular cylinder, centered on } z\text{-axis, radius } 2}$



(d) $2x^2 + 8y^2 + z^2 = 16$

Cross sections are all ellipses. $\boxed{\text{Ellipsoid centered at } (0,0,0)}$

10. (10 points) Let $G(x, y, z) = \sqrt{x^2 + y^2 - z}$.

(a) Evaluate $G(-4, 5, -8)$.

$$G(-4,5,-8) = \sqrt{16+25+8} = \sqrt{49} = \boxed{7}$$

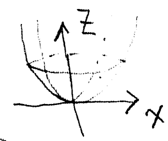
(b) What is the domain of G?

MUST HAVE $x^2+y^2-z \geq 0$

OR $z \leq x^2+y^2$

$$\boxed{D = \{(x,y,z) : z \leq x^2+y^2\}}$$

(c) Describe or sketch the level surface $G(x, y, z) = 0$.



$G(x,y,z) = 0 \Rightarrow z = x^2+y^2$

THE LEVEL SURFACE IS A CIRCULAR PARABOLOID WITH VERTEX AT $(0,0,0)$ AND OPENING UP THE Z-AXIS

(d) Describe or sketch the level surface $G(x, y, z) = -1$.

$G(x,y,z) = -1 \Rightarrow \sqrt{x^2+y^2-z} = -1$

NO WAY! NO SUCH LEVEL CURVE. -1 IS NOT IN THE RANGE OF G.

(e) Describe or sketch the level surface $G(x, y, z) = 1$.

$G(x,y,z) = 1 \Rightarrow x^2+y^2-z = 1$ OR $\underbrace{x^2+y^2-1 = z}$

SAME PARABOLOID AS IN (c) BUT SHIFTED DOWN Z-AXIS ONE UNIT.

11. (5 points) Find the curvature function for $y = e^x$. What happens to the curvature as $x \to \infty$? Explain how/why your answer is obvious from the graph of $y = e^x$.

$y = e^x$

$\frac{dy}{dx} = e^x$

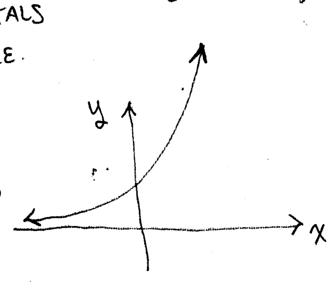
$\frac{d^2y}{dx^2} = e^x$

$$\boxed{K(x) = \frac{e^x}{(1+e^{2x})^{3/2}}}$$

THE GRAPH IS PRETTY STRAIGHT AS X GETS BIG.

$$\lim_{x\to\infty} \frac{e^x}{(1+e^{2x})^{3/2}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{e^x}{\frac{3}{2}(1+e^{2x})^{1/2}(2)e^{2x}} = \lim_{x\to\infty} \frac{1}{3e^x(1+e^{2x})^{1/2}} = 0$$

L'HOPITAL'S RULE.



$$\boxed{\lim_{x\to\infty} K(x) = 0}$$