Math 233 - Test 3

November 9, 2023

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (14 points) Determine the limit or show that it does not exist.

   (a) $\lim_{(x,y)\to(1,1)} \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(1,1)} \frac{y\cancel{(y-x)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}\ (-1)}$

   $= \lim_{(x,y)\to(1,1)} (-1)(y)(\sqrt{x}+\sqrt{y}) = \boxed{-2}$

   (b) $\lim_{(x,y)\to(0,2)} \frac{x^2(y-2)}{x^4 + (y-2)^2}$

   Along $x = 0$: $\lim_{y\to 2} \frac{0}{(y-2)^2} = 0$

   Along $y = x^2 + 2$: $\lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

   Two limits along two paths

   $\Downarrow$

   $\boxed{\text{Limit DNE.}}$

2. (2 points) For the function F, under what conditions would you expect $F_{xyz}(x, y, z) = F_{yzx}(x, y, z)$?

   F_{xyz} and F_{yzx} are continuous in a neighborhood of (x,y,z), then they'll be equal.

3. (7 points) Show that f is not continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} = \underbrace{\cos^2\theta - \sin^2\theta}_{\text{DEPENDS ON } \theta}$$

LIMIT DNE BY TWO-PATH TEST.

LIMIT AT (0,0) DNE

$\Rightarrow$ f CANNOT BE CONT. AT (0,0)

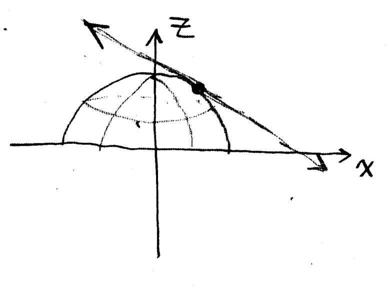
4. (8 points) Let $u = e^{-2t} \sin x \sin y$. Show that $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

$$\frac{\partial u}{\partial t} = -2e^{-2t} \sin x \sin y, \quad \frac{\partial u}{\partial x} = e^{-2t} \cos x \sin y, \quad \frac{\partial u}{\partial y} = e^{-2t} \sin x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-2t}(-\sin x)\sin y, \quad \frac{\partial^2 u}{\partial y^2} = e^{-2t} \sin x(-\sin y)$$

$$-2e^{-2t} \sin x \sin y = -e^{-2t} \sin x \sin y + (-e^{-2t} \sin x \sin y) \checkmark$$

5. (3 points) The graph of $g(x,y) = \sqrt{1-x^2-y^2}$ is the upper half of the unit sphere centered at the origin. Think about the graph at the point $(1/2, 0)$. Without actually computing it, tell me the sign of $g_x(1/2,0)$ and say how you know.



$g_x(\frac{1}{2},0)$ = SLOPE OF GRAPH IN THE DIRECTION OF POS. X-AXIS. = NEGATIVE

SEE GRAPH. LOOKING DOWN POS X-AXIS AT $(\frac{1}{2},0)$, YOU WOULD BE LOOKING DOWNHILL.

6. (8 points) Find the linearization of $f(x, y) = \sqrt{41 - 4x^2 - y^2}$ at $(x, y) = (2, 3)$. Then use your linearization to approximate $f(2.1, 2.9)$.

$$f(2,3) = \sqrt{41 - 16 - 9} = \sqrt{16} = 4$$

$$f_x(x,y) = \frac{-4x}{\sqrt{41-4x^2-y^2}}, \quad f_x(2,3) = -\frac{8}{4}$$

$$f_y(x,y) = \frac{-y}{\sqrt{41-4x^2-y^2}}, \quad f_y(2,3) = -\frac{3}{4}$$

$$\boxed{L(x,y) = 4 - 2(x-2) - \frac{3}{4}(y-3)}$$

$$\rightarrow f(2.1, 2.9) \approx L(2.1, 2.9) = 4 - 0.2 + 0.075 = 3.875$$

$$\boxed{f(2.1, 2.9) \approx 3.875}$$

7. (5 points) Suppose w is a function of x, y, z and x, y, z are functions of s, t. Write the chain rule formulas for $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

8. (5 points) Assume that y is implicitly defined as a function of x by the equation $xe^y + ye^x = 2x^2y$. Use partial derivatives to find dy/dx.

LET $F(x,y) = xe^y + ye^x - 2x^2y$.

THEN $\boxed{\dfrac{dy}{dx} = \dfrac{-F_x}{F_y} = \dfrac{-(e^y + ye^x - 4xy)}{xe^y + e^x - 2x^2}}$

9. (6 points) The electric voltage in a certain region in space is described by the function $V(x, y, z) = 5x^2 - 3xy + xyz$. At the point $(3, 4, 5)$, in what direction is the voltage increasing most rapidly? What is the rate of change in that direction?

DIRECTION OF $\vec{\nabla} V(3,4,5)$.

$$\vec{\nabla} V(x,y,z) = (10x - 3y + yz)\,\hat{i} + (-3x + xz)\,\hat{j} + (xy)\,\hat{k}$$

$$\vec{\nabla} V(3,4,5) = \boxed{38\hat{i} + 6\hat{j} + 12\hat{k}} = \text{DIRECTION OF MAX INCREASE}$$

RATE OF CHANGE IN THAT DIRECTION $= \|\vec{\nabla} V(3,4,5)\|$

$$= \sqrt{38^2 + 6^2 + 12^2} = \boxed{\begin{array}{c}\sqrt{1624} \\ \approx 40.3\end{array}}$$

10. (8 points) Identify each quadric surface.

(a) $5x^2 - 4y^2 + 20z^2 = 0$

HYPERBOLAS / ELLIPSES / HYPERBOLAS , LINES AT $z = 0$

CONE

(b) $-x^2 + 36y^2 + 36z^2 = 9$

ELLIPSES / HYPERBOLAS / HYPERBOLAS, ELLIPES FOR ALL X

$36y^2 + 36z^2 = 9 + x^2$

HYPERBOLOID OF ONE SHEET

(c) $z - 4x^2 + y^2 = 0$

PARABOLAS / PARABOLAS / HYPERBOLAS

HYPERBOLIC PARABOLOID

(d) $-3x^2 + 5y^2 - z^2 = 10$

HYPERBOLAS / ELLIPSES / HYPERBOLAS, NO ELLIPSES FOR SMALL Z

$5y^2 - 10 = 3x^2 + z^2$

HYPERBOLOID OF TWO SHEETS

11. (6 points) Find an equation of the plane tangent to the surface $x^3 + y^3 = 3xyz$ at the point $(1, 2, 3/2)$.

$F(x,y,z) = x^3 + y^3 - 3xyz$

OUR SURFACE IS THE LEVEL SURFACE

$F(x,y,z) = 0.$

$\vec{\nabla} F(x,y,z) = (3x^2 - 3yz)\,\hat{\imath} + (3y^2 - 3xz)\,\hat{\jmath}$
$- 3xy\,\hat{k}$

$\vec{\nabla} F(1,2,3/2) = -6\hat{\imath} + \frac{15}{2}\hat{\jmath} - 6\hat{k}$

TANGENT PLANE IS $-6(x-1) + \frac{15}{2}(y-2) - 6\left(z - \frac{3}{2}\right) = 0$

-OR- $-6x + \frac{15}{2}y - 6z = 0$

12. (8 points) Find the critical points of the function $f(x, y) = x^3 - 3xy + y^2 + y - 5$.

$f_x(x,y) = 3x^2 - 3y = 0 \Rightarrow y = x^2$

$f_y(x,y) = -3x + 2y + 1 = 0$

$2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$x = \frac{1}{2}, \quad x = 1$

$y = \frac{1}{4} \qquad y = 1$

TWO CRIT PTS: $\left(\frac{1}{2}, \frac{1}{4}\right)$ AND $(1,1)$