Math 233 - Final Exam B

December 14, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Determine the limit or show that it does not exist.

(a) $\lim\limits_{(x,y)\to(4,5)} \dfrac{\sqrt{x+y}-3}{x+y-9} \cdot \dfrac{\sqrt{x+y}+3}{\sqrt{x+y}+3} = \lim\limits_{(x,y)\to(4,5)} \dfrac{\cancel{x+y-9}}{\cancel{(x+y-9)}(\sqrt{x+y}+3)}$

$$= \frac{1}{\sqrt{4+5}+3} = \boxed{\frac{1}{6}}$$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{xy^2}$

Along $y=x$:

$$\lim_{x\to 0} \frac{2x^3}{x^3} = \lim_{x\to 0} 2 = 2$$

Along $y=2x$:

$$\lim_{x\to 0} \frac{x^3+8x^3}{4x^3} = \lim_{x\to 0} \frac{9}{4} = \frac{9}{4}$$

$$2 \neq \frac{9}{4}$$

LIMIT DNE BY TWO-PATH TEST

2. (10 points) For $t>0$, let $\vec{r}(t) = t^2\,\hat{\imath} + 4t\,\hat{\jmath} + \ln t^4\,\hat{k}$. Compute the unit tangent vector $\hat{T}(t)$. Then compute $\hat{T}(2)$ and show that it indeed has magnitude 1.

$$\vec{r}\,'(t) = 2t\hat{\imath} + 4\hat{\jmath} + \frac{4}{t}\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2+16+\frac{16}{t^2}}$$

$$= \sqrt{\frac{4t^4+16t^2+16}{t^2}}$$

$$= \frac{2t^2+4}{t}, \quad t>0$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{t}{2t^2+4}\,\vec{r}\,'(t)$$

$$\boxed{\hat{T}(t) = \frac{1}{t^2+2}\left(t^2\hat{\imath} + 2t\hat{\jmath} + 2\hat{k}\right)}$$

Follow-up:

$$\hat{T}(2) = \frac{1}{6}\left(4\hat{\imath} + 4\hat{\jmath} + 2\hat{k}\right)$$

$$\|\hat{T}(2)\| = \frac{1}{6}\sqrt{16+16+4} = \frac{1}{6}\sqrt{36}$$

$$= 1 \checkmark$$

3. (10 points) Consider the surface described by the equation

$$\sin z = \frac{1}{xy}.$$

Let $F(x,y,z) = xy \sin z$

Our surface is the level surface $F(x,y,z) = 1$

(a) Find an equation of the plane tangent to the surface at the point $(1, 2, \pi/6)$.

$$\vec{\nabla} F(x,y,z) = y \sin z \,\hat{\imath} + x \sin z \,\hat{\jmath} + xy \cos z \,\hat{k}$$

$$\vec{n} = \vec{\nabla} F(1,2,\pi/6) = \hat{\imath} + \tfrac{1}{2}\hat{\jmath} + \sqrt{3}\,\hat{k}$$

$$\boxed{\text{Tan plane: } (x-1) + \tfrac{1}{2}(y-2) + \sqrt{3}\left(z - \tfrac{\pi}{6}\right) = 0 \quad \text{or } x + \tfrac{1}{2}y + \sqrt{3}z = 2 + \tfrac{\sqrt{3}\pi}{6}}$$

(b) Find a set of parametric equations for the line normal to the surface at the point $(1, 2, \pi/6)$.

$$\boxed{\begin{aligned} x &= t + 1 \\ y &= \tfrac{1}{2}t + 2 \\ z &= \sqrt{3}\,t + \tfrac{\pi}{6} \end{aligned}}$$

4. (10 points) Find the critical points of $f(x,y) = x^2 - \frac{1}{2}x^4 - y^2 - xy$. Then use the second partials test to classify the critical points and find the extreme values.

$$f_x(x,y) = 2x - 2x^3 - y = 0$$

$$f_y(x,y) = -2y - x = 0$$

$$x = -2y$$

$$16y^3 - 5y = 0$$

$$y(16y^2 - 5) = 0$$

$$y = 0, \quad y = \pm\frac{\sqrt{5}}{4}$$

$$\downarrow \qquad \downarrow$$

$$x = 0 \qquad x = \mp\frac{\sqrt{5}}{2}$$

$$\boxed{\text{Crit pts: } (0,0),\ \left(-\tfrac{\sqrt{5}}{2}, \tfrac{\sqrt{5}}{4}\right),\ \left(\tfrac{\sqrt{5}}{2}, -\tfrac{\sqrt{5}}{4}\right)}$$

$$D(x,y) = \begin{vmatrix} 2-6x^2 & -1 \\ -1 & -2 \end{vmatrix} = 12x^2 - 5$$

$(0,0)$ ... $D(0,0) = -5$

$\Rightarrow$ $\boxed{(0,0,0) \text{ is a saddle pt.}}$

$\left(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{4}\right)$ ... $D\left(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{4}\right) = 10 > 0$

AND $f_{yy}\left(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{4}\right) < 0$

$\Rightarrow$ $\boxed{f\left(-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{4}\right) = \frac{25}{32} = 0.78125 \text{ is a rel. max.}}$

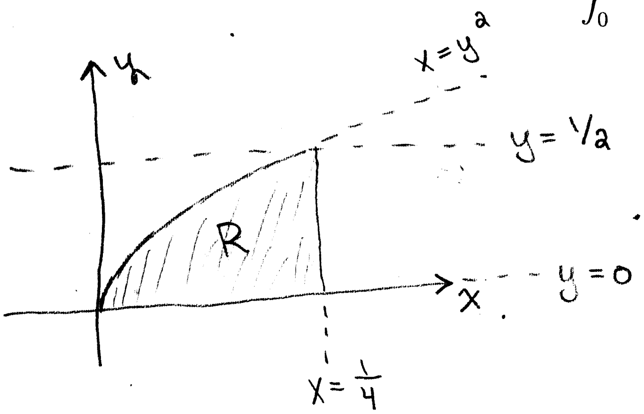
$\left(\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{4}\right)$ ... $D\left(\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{4}\right) = 10 > 0$

AND $f_{yy}\left(\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{4}\right) < 0$

$\Rightarrow$ $\boxed{f\left(\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{4}\right) = \frac{25}{32} = 0.78125 \text{ is a rel. max.}}$

5. (10 points) Sketch the region of integration, reverse the order of integration, and evaluate.

$$\int_0^{1/2} \int_{y^2}^{1/4} y \cos(16\pi x^2)\, dx\, dy$$



$$\int_{x=0}^{x=\frac{1}{4}} \int_{y=0}^{y=\sqrt{x}} y \cos(16\pi x^2)\, dy\, dx$$

$$= \int_0^{1/4} \frac{x}{2} \cos(16\pi x^2)\, dx$$

$u = 16\pi x^2 \quad du = 32\pi x\, dx$

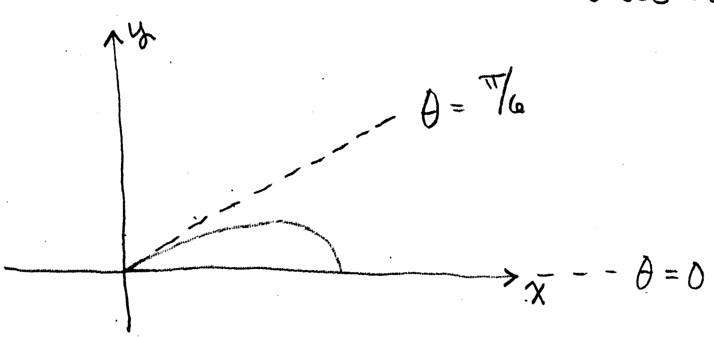
$\frac{1}{32\pi} du = x\, dx$

$$= \frac{1}{64\pi} \int_{u=0}^{u=\pi} \cos u\, du = \frac{1}{64\pi} \sin(u) \Big|_0^{\pi} = \boxed{0}$$

6. (10 points) Find the total area of the three petals of the rose curve $r = 2\cos 3\theta$. (Take advantage of symmetry! And, you may need to use a power reducing formula when integrating.)

$2\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}$



THERE ARE 3 PETALS.

THAT ↑ IS HALF OF THE "FIRST" ONE.

TOTAL AREA =

$$6 \int_{\theta=0}^{\theta=\pi/6} \int_{r=0}^{r=2\cos 3\theta} r\, dr\, d\theta$$

$$= 6 \int_0^{\pi/6} \frac{4\cos^2 3\theta}{2}\, d\theta =$$

$$12 \int_0^{\pi/6} \cos^2 3\theta\, d\theta = 12 \int_0^{\pi/6} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta\right) d\theta$$

$$= 6\theta + \sin 6\theta \Big|_0^{\pi/6} = \boxed{\pi}$$

7. (10 points) Evaluate $\int_C (xy+2z)\,ds$, where C is the line segment in space from $P\,(0,1,1)$ to $Q\,(1,0,0)$.

$\vec{PQ} = \hat{i} - \hat{j} - \hat{k} \Rightarrow \quad C: \; x = t, \; y = -t+1, \; z = -t+1; \; 0 \le t \le 1$

$$\vec{r}(t) = t\hat{i} + (-t+1)\hat{j} + (-t+1)\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$$

$$\int_C (xy+2z)\,ds = \int_0^1 [t(-t+1) + 2(-t+1)]\sqrt{3}\,dt$$

$$= \int_0^1 (-t^2 - t + 2)\sqrt{3}\,dt$$

$$= \sqrt{3}\left(-\frac{1}{3}t^3 - \frac{1}{2}t^2 + 2t\right)\Big|_0^1$$

$$= \sqrt{3}\left(-\frac{1}{3} - \frac{1}{2} + 2\right)$$

$$= \boxed{\frac{7\sqrt{3}}{6}}$$