Math 233 - Test 3
November 6, 2025

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (3 points) The function f is defined below. It is continuous everywhere, but it can be shown that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Think about our theorem regarding the equality of mixed partial derivatives. Explain what conclusion must be drawn about f_{xy} or f_{yx}.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

IF f_{xy} AND f_{yx} ARE CONTINUOUS IN A NEIGHBORHOOD OF $(0,0)$, THEY WOULD BE EQUAL AT $(0,0)$.

$\boxed{\text{CONCLUSION: } f_{xy} \text{ OR } f_{yx} \text{ IS NOT CONTINUOUS IN A NEIGHBORHOOD OF } (0,0).}$

2. (6 points) Let $f(x,y) = \tan^{-1}(\frac{y}{x})$. Evaluate f_x and f_y at the point $(2,-2)$.

$$\boxed{f_x(x,y) = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right)} \qquad f_x(2,-2) = \frac{1}{2}\left(\frac{2}{4}\right) = \frac{1}{4}$$

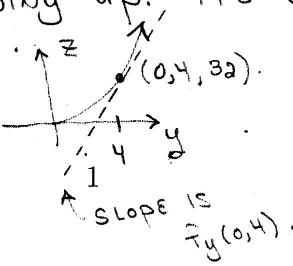
$$\boxed{f_x(2,-2) = \frac{1}{4}}$$

$$\boxed{f_y(x,y) = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right)} \qquad f_y(2,-2) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\boxed{f_y(2,-2) = \frac{1}{4}}$$

3. (4 points) Think about the graph of the function $f(x,y) = 5x^2 + 2y^2$. Without computing any derivatives, explain how/why we should know that $f_y(0,4)$ is positive.

THE GRAPH OF $f(x,y)$ IS AN ELLIPTIC PARABOLOID WITH VERTEX AT $(0,0,0)$ AND OPENING UP. ITS CROSS SECTION IN THE $X=0$ PLANE LOOKS LIKE



TANGENT LINE HAS POSITIVE SLOPE.

4. (10 points)

(a) Determine the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

0% MORE WORK.

Along $x = 0$...

$\lim_{y\to 0} \frac{0}{y^2} = \underline{\underline{0}}$

Along $y = x^2$...

$\lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$

LIMIT DNE BY TWO-PATH TEST.

(b) Is the following function continuous at $(0,0)$? Explain.

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

No way!

$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0$

BECAUSE THE LIMIT DNE.

5. (10 points) Determine the limit or show that it does not exist.

(a) $\lim_{(x,y)\to(3,-1)} \frac{x^2+2xy-3y^2}{x^2+9y^2} = \frac{9-6-3}{9+9} = \frac{0}{18} = \boxed{0}$

(b) $\lim_{(x,y)\to(1,1)} \frac{y^2-xy}{\sqrt{x}-\sqrt{y}}$

0% MORE WORK.

$\lim_{(x,y)\to(1,1)} \frac{y(y-x)}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(1,1)} \frac{y\overset{-1}{\cancel{(y-x)}}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$

$= \lim_{(x,y)\to(1,1)} -y(\sqrt{x}+\sqrt{y})$

$= \boxed{-2}$

6. (7 points) The equation below is called *Laplace's equation.* It is an example of a partial differential equation that arises in some very important applications.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Show that $\underbrace{z = \ln(\sqrt{x^2+y^2})}$ satisfies Laplace's equation.

$z = \frac{1}{2}\ln(x^2+y^2)$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= \frac{x}{x^2+y^2}, \quad \frac{\partial^2 z}{\partial x^2} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \\ \frac{\partial z}{\partial y} &= \frac{y}{x^2+y^2}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{-y^2+x^2}{(x^2+y^2)^2} \end{aligned}\right\} \text{THEIR SUM IS } 0.$$

7. (8 points) Find the linearization of $f(x,y) = \dfrac{xy}{x+y}$ at $(x,y) = (-1,2)$. Then use your linearization to approximate $f(-0.9, 1.9)$.

$f(-1,2) = -2$

$f_x(x,y) = \dfrac{y(x+y) - xy}{(x+y)^2}$

$f_x(-1,2) = \dfrac{4}{1} = 4$

$f_y(x,y) = \dfrac{x(x+y) - xy}{(x+y)^2}$

$f_y(-1,2) = 1$

$\boxed{L(x,y) = -2 + 4(x+1) + 1(y-2)}$

$f(-0.9, 1.9) \approx L(-0.9, 1.9) = -2 + 0.4 - 0.1$
$= \boxed{-1.7}$

8. (6 points) Use differentials to approximate the change in the area of a triangle if its base is increased from 5 cm to 5.1 cm and its height is decreased from 10 cm to 9.8 cm.

$A = \frac{1}{2}bh$

$h = 10\text{ cm} \qquad \Delta h = -0.2\text{ cm}$

$b = 5\text{ cm} \qquad \Delta b = 0.1\text{ cm}$

$\Delta A \approx A_b \Delta b + A_h \Delta h$

$= \frac{1}{2}h\Delta b + \frac{1}{2}b\Delta h$

$\frac{1}{2}(10)(0.1) + \frac{1}{2}(5)(-0.2)$

$= \boxed{0\text{ cm}^2}$

$$\frac{\partial z}{\partial x} = 6-2x, \quad \frac{\partial z}{\partial y} = -4-4y$$

9. (6 points) Find all points on the surface $z = 6x - 4y - x^2 - 2y^2$ at which the tangent plane is parallel to the xy-plane. (Hint: For any such tangent plane, the normal vector must be parallel to $\hat{k}$.)

LINEARIZATION AT (a,b) IS

$$\underbrace{L(x,y) = f(a,b) + (6-2a)(x-a) + (-4-4b)(y-b)}$$

NORMAL VECTOR IS $(6-2a)\hat{\imath} + (-4-4b)\hat{\jmath} - \hat{k}$

PARALLEL TO $\hat{k}$

$\Rightarrow \quad 6-2a = 0$

$-4-4b = 0$

$\boxed{(a,b) = (3,-1)}$

$f(3,-1) = 11$

$\boxed{\text{POINT ON GRAPH } (3,-1,11)}$

10. (8 points) Suppose $z = f(x,y)$, where $x = g(s,t)$ and $y = h(s,t)$. Also assume that all of these functions are differentiable.

(a) Write the chain rule formula for $\dfrac{\partial z}{\partial s}$.

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial z}}$$

(b) Now use the following information to determine $\dfrac{\partial z}{\partial s}$ at $(s,t) = (2,-1)$.

$$\frac{\partial x}{\partial s} = 5, \quad \frac{\partial y}{\partial s} = 2, \quad g(2,-1) = 3, \quad h(2,-1) = 4, \quad f_x(3,4) = 12, \quad f_y(3,4) = 7$$

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t) = (2,-1)} = \left.\frac{\partial z}{\partial x}\right|_{(s,t)=(2,-1)} \left.\frac{\partial x}{\partial s}\right|_{(s,t)=(2,-1)} + \left.\frac{\partial z}{\partial y}\right|_{(s,t)=(2,1)} \left.\frac{\partial y}{\partial s}\right|_{(s,t)=(2,1)}$$

$$= \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(3,4)} (5) + \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(3,4)} (2)$$

$$= (12)(5) + (7)(2) = \boxed{74}$$

11. (6 points) Suppose that y is implicitly defined as a function of x and z by the equation

$$\underbrace{3x^2z - x^2y^2 + 2z^3 + 3yz - 5}_{F(x,y,z)} = 0.$$

Find $\partial y/\partial z$.

$$\frac{\partial y}{\partial z} = \frac{-F_z}{F_y} = \frac{-(3x^2 + 6z^2 + 3y)}{-2x^2y + 3z}$$

12. (8 points) Identify each quadric surface.

(a) $4x^2 - 18y^2 + 9z^2 = 36$

Fix $x=k$: Hyperbolas

Fix $y=k$: Ellipses → Ellipses, $4x^2+9z^2=36+18k^2$, exist for any k.

Fix $z=k$: Hyperbolas

$\boxed{\text{Hyperboloid of one sheet.}}$

(b) $4y^2 + 9z^2 - 36x^2 = 0$

Fix $x=k$: Ellipses → Ellipses, $4y^2+9z^2=36k^2$ degenerate to a point at $(0,0,0)$.

Fix $y=k$: Hyperbolas

Fix $z=k$: Hyperbolas

$\boxed{\text{Cone}}$

(c) $x^2 + z^2 - 4y = 0$

$4y = x^2+z^2$

$y = \frac{1}{4}(x^2+z^2) \rightarrow$ $\boxed{\text{Circular paraboloid}}$ opening up pos. y. axis.

(d) $3x^2 + 2y^2 - z^2 + 1 = 0$

Fix $x=k$: Hyperbolas

Fix $y=k$: Hyperbolas

Fix $z=k$: Ellipses → Ellipses, $3x^2+2y^2=k^2-1$, exist only for $|k|>1$.

$\boxed{\text{Hyperboloid of two sheets}}$

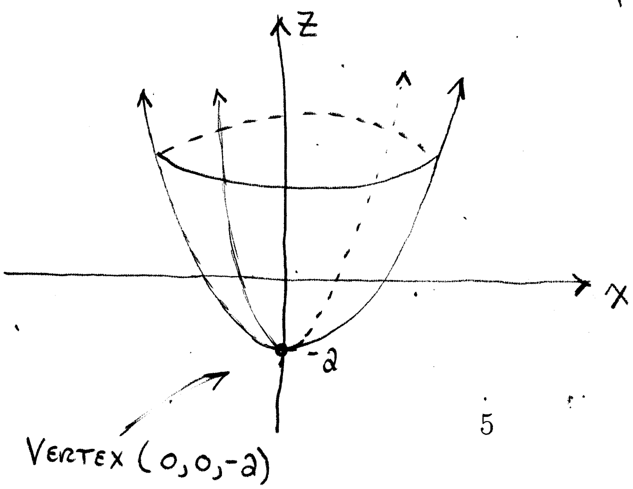
13. (6 points) Identify the quadric surface, describe it, and sketch its graph. (Your graph should show some of the level curves.)

$$z + 2 = 2x^2 + 2y^2$$

$$z = -2 + 2(x^2+y^2)$$

Circular paraboloid. Vertex at $(0,0,-2)$.

Opening up z-axis.



Parabolas in planes parallel to yz-plane.

Parabolas in planes parallel to xz-plane.

Circles, above $z=-2$, in planes parallel to xy-plane.

14. (8 points) The temperature at the point (x, y) on a metal plate is given by

$$T(x, y) = e^x(\sin 2x + \sin 3y).$$

(a) Find the directional derivative of T at the point $P\,(0, 0)$ in the direction from P to $Q\,(1, 3)$. Round your final answer to the nearest hundredth.

$\vec{PQ} = \langle 1, 3 \rangle$

$\|\vec{PQ}\| = \sqrt{10}$

$\vec{u} = \dfrac{\vec{PQ}}{\|\vec{PQ}\|} = \left\langle \dfrac{1}{\sqrt{10}}, \dfrac{3}{\sqrt{10}} \right\rangle$

$T_x(x,y) = e^x(\sin 2x + \sin 3y) + 2e^x \cos 2x \Rightarrow T_x(0,0) = 2$

$T_y(x,y) = 3e^x \cos 3y \Rightarrow T_y(0,0) = 3$

$D_{\vec{u}}\, T(0,0) = \dfrac{2}{\sqrt{10}} + \dfrac{9}{\sqrt{10}} = \boxed{\dfrac{11}{\sqrt{10}} \approx 3.48}$

(b) Is the temperature increasing or decreasing in the direction from P to Q? Explain.

POSITIVE DIRECTIONAL DERIVATIVE

$\Rightarrow$ T IS INCREASING IN DIRECTION OF $\vec{u}$.

15. (4 points) The *gradient* of the function $f(x, y)$ is the vector defined by

$$\vec{\nabla} f(x, y) = f_x(x, y)\,\hat{\imath} + f_y(x, y)\,\hat{\jmath}.$$

Compute $\vec{\nabla} f(x, y)$ when $f(x, y) = x^2y - xy^3 - 5x + y$.

$f_x(x,y) = 2xy - y^3 - 5$

$f_y(x,y) = x^2 - 3xy^2 + 1$

$$\boxed{\vec{\nabla} f(x,y) = (2xy - y^3 - 5)\,\hat{\imath} + (x^2 - 3xy^2 + 1)\,\hat{\jmath}}$$