Math 233 - Test 3

November 6, 2025

Name ______________________

Score __________

Show all work to receive full credit. Supply explanations where necessary.

1. (3 points) The function f is defined below. It is continuous everywhere, but it can be shown that $f_{xy}(0,0) \neq f_{yx}(0,0)$. Think about our theorem regarding the equality of mixed partial derivatives. Explain what conclusion must be drawn about f_{xy} or f_{yx}.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

2. (6 points) Let $f(x,y) = \tan^{-1}(\frac{y}{x})$. Evaluate f_x and f_y at the point $(2,-2)$.

3. (4 points) Think about the graph of the function $f(x,y) = 5x^2 + 2y^2$. Without computing any derivatives, explain how/why we should know that $f_y(0,4)$ is positive.

4. (10 points)

(a) Determine the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

(b) Is the following function continuous at $(0,0)$? Explain.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

5. (10 points) Determine the limit or show that it does not exist.

(a) $\displaystyle \lim_{(x,y)\to(3,-1)} \frac{x^2 + 2xy - 3y^2}{x^2 + 9y^2}$

(b) $\displaystyle \lim_{(x,y)\to(1,1)} \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$

6. (7 points) The equation below is called *Laplace's equation.* It is an example of a partial differential equation that arises in some very important applications.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Show that $z = \ln(\sqrt{x^2 + y^2})$ satisfies Laplace's equation.

7. (8 points) Find the linearization of $f(x, y) = \dfrac{xy}{x + y}$ at $(x, y) = (-1, 2)$. Then use your linearization to approximate $f(-0.9, 1.9)$.

8. (6 points) Use differentials to approximate the change in the area of a triangle if its base is increased from 5 cm to 5.1 cm and its height is decreased from 10 cm to 9.8 cm.

9. (6 points) Find all points on the surface $z = 6x - 4y - x^2 - 2y^2$ at which the tangent plane is parallel to the xy-plane. (Hint: For any such tangent plane, the normal vector must be parallel to $\hat{k}$.)

10. (8 points) Suppose $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$. Also assume that all of these functions are differentiable.

    (a) Write the chain rule formula for $\dfrac{\partial z}{\partial s}$.

    (b) Now use the following information to determine $\dfrac{\partial z}{\partial s}$ at $(s, t) = (2, -1)$.

$$\frac{\partial x}{\partial s} = 5, \quad \frac{\partial y}{\partial s} = 2, \quad g(2, -1) = 3, \quad h(2, -1) = 4, \quad f_x(3, 4) = 12, \quad f_y(3, 4) = 7$$

11. (6 points) Suppose that y is implicitly defined as a function of x and z by the equation

$$3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0.$$

Find $\partial y/\partial z$.

12. (8 points) Identify each quadric surface.

(a) $4x^2 - 18y^2 + 9z^2 = 36$

(b) $4y^2 + 9z^2 - 36x^2 = 0$

(c) $x^2 + z^2 - 4y = 0$

(d) $3x^2 + 2y^2 - z^2 + 1 = 0$

13. (6 points) Identify the quadric surface, describe it, and sketch its graph. (Your graph should show some of the level curves.)

$$z + 2 = 2x^2 + 2y^2$$

14. (8 points) The temperature at the point (x, y) on a metal plate is given by

$$T(x, y) = e^x(\sin 2x + \sin 3y).$$

(a) Find the directional derivative of T at the point $P\,(0, 0)$ in the direction from P to $Q\,(1, 3)$. Round your final answer to the nearest hundredth.

(b) Is the temperature increasing or decreasing in the direction from P to Q? Explain.

15. (4 points) The *gradient* of the function $f(x, y)$ is the vector defined by

$$\vec{\nabla} f(x, y) = f_x(x, y)\,\hat{\imath} + f_y(x, y)\,\hat{\jmath}.$$

Compute $\vec{\nabla} f(x, y)$ when $f(x, y) = x^2y - xy^3 - 5x + y$.