# Math 233 - Final Exam A

December 5, 2025

Name key

Score ________

**Show all work to receive full credit**. Supply explanations where necessary. This portion of the test is due December 11. You must work individually. All integration must be done by hand, but you may use technology to check your answers.

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1. (10 points) Let $f(x,y) = 2xy^2 - x^2y + 4xy$. Determine all relative maxima, relative minima, and saddle points.

$f_x(x,y) = 2y^2 - 2xy + 4y = 0 \Rightarrow 2y(y - x + 2) = 0$

$y = 0$ OR $y = x - 2$

$f_y(x,y) = 4xy - x^2 + 4x = 0$

$y = 0 \rightarrow -x^2 + 4x = 0 \rightarrow x = 0,\ x = 4$

$y = x - 2 \rightarrow 4x^2 - 8x - x^2 + 4x = 0$

$3x^2 - 4x = 0$

$x = 0, \quad x = \frac{4}{3}$

$y = -2 \quad y = -\frac{2}{3}$

4 CRITICAL PTS:

$(0,0),\ (4,0),\ (0,-2),\ \left(\frac{4}{3}, -\frac{2}{3}\right)$

$$D(x,y) = \begin{vmatrix} -2y & 4y - 2x + 4 \\ 4y - 2x + 4 & 4x \end{vmatrix}$$

$$= -8xy - (4y - 2x + 4)^2$$

$D(0,0) = -16 \Rightarrow$ $(0,0,0)$ IS A SADDLE POINT.

$D(4,0) = -16 \Rightarrow$ $(4,0,0)$ IS A SADDLE POINT.

$D(0,-2) = -16 \Rightarrow$ $(0,-2,0)$ IS A SADDLE POINT.

$D\left(\frac{4}{3}, -\frac{2}{3}\right) = \frac{16}{3}$ AND

$f_{xx}\left(\frac{4}{3}, -\frac{2}{3}\right) = \frac{4}{3} > 0 \Longrightarrow$ $f\left(\frac{4}{3}, -\frac{2}{3}\right) = -\frac{32}{27}$ IS A RELATIVE MINIMUM

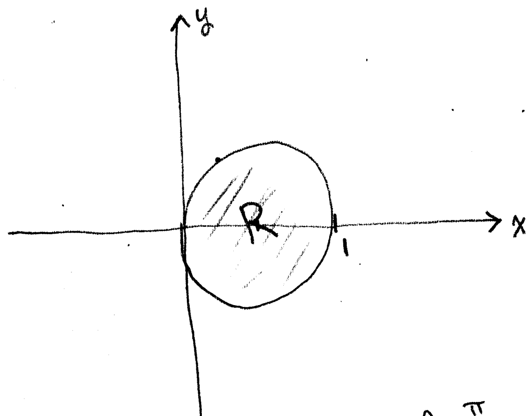
2. (10 points) Use a double integral in polar coordinates to find the volume of the solid that lies inside the cylinder $x^2 + y^2 - x = 0$ and is bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the $xy$-plane.

$$x^2+y^2-x=0 \Rightarrow \left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$$

CIRCLE CENTERED AT $\left(\frac{1}{2}, 0\right)$ w/ RADIUS $\frac{1}{2}$.

$$x^2+y^2=x \Rightarrow r^2=r\cos\theta$$

$$\Rightarrow r=\cos\theta$$




$$\text{Volume} = \iint_R (1-x^2-y^2)\, dA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} (1-r^2)\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left. \frac{1}{2}r^2 - \frac{1}{4}r^4 \right|_0^{\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\cos^2\theta - \frac{1}{4}\cos^4\theta\right) d\theta$$

$\}$ POWER REDUCING FORMULAS.

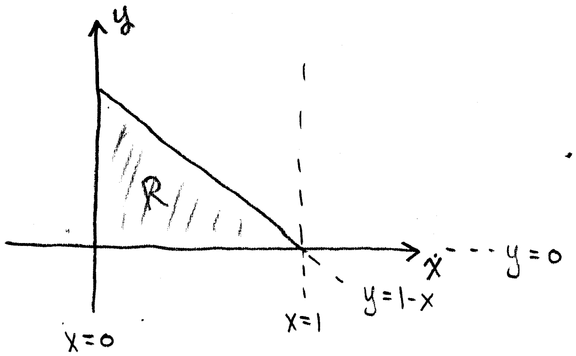
$$= \int_{-\pi/2}^{\pi/2} \left[\frac{1}{4}(1+\cos 2\theta) - \frac{1}{32}(3+4\cos 2\theta + \cos 4\theta)\right] d\theta$$

$$= \left. \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{3}{32}\theta - \frac{1}{16}\sin 2\theta - \frac{1}{128}\sin 4\theta \right|_{-\pi/2}^{\pi/2}$$



$$= \frac{1}{4}\left(\frac{\pi}{2}\right) - \frac{3}{32}\left(\frac{\pi}{2}\right) + \frac{1}{4}\left(\frac{\pi}{2}\right) - \frac{3}{32}\left(\frac{\pi}{2}\right) = \boxed{\frac{5\pi}{32}}$$

3. (10 points) The base of a triangular prism lies in the $xy$-plane bounded by the graphs of $y = 0$, $x = 0$, and $y = 1 - x$. The top of the prism lies in the plane $2x + y + z = 4$. The density of the prism at the point $(x, y, z)$ is given by $\rho(x, y, z) = x + 2$. Use a triple integral to find the mass of the prism.




$$\text{Mass} = \iint_R \int_{z=0}^{z=4-2x-y} (x+2)\, dz\, dA$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=4-2x-y} (x+2)\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (x+2)(4-2x-y)\, dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (\cancel{4x} - 2x^2 - xy + 8 - \cancel{4x} - 2y)\, dy\, dx$$

$$= \int_0^1 \left(-2x^2 y - \tfrac{1}{2} x y^2 + 8y - y^2\right) \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \underbrace{\left[-2x^2(1-x) - \tfrac{1}{2} x (1-x)^2 + 8(1-x) - (1-x)^2\right]}_{-2x^2 + 2x^3 - \frac{1}{2}x + x^2 - \frac{1}{2}x^3 + 8 - 8x - 1 + 2x - x^2} dx$$

$$= \int_0^1 \left(\tfrac{3}{2} x^3 - 2x^2 - \tfrac{13}{2} x + 7\right) dx = \frac{3}{8} - \frac{2}{3} - \frac{13}{4} + 7 = \boxed{\frac{83}{24}}$$