Math 233 - Final Exam B

December 11, 2025

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Let $\vec{r}(t) = (8-t^2)\,\hat{\imath} + t^3\,\hat{\jmath} + \frac{\sqrt{5}}{2}t^2\,\hat{k}$. Starting from $t = 0$, find the arc-length parameter $s(t)$, and then use the arc-length parameter to find the length of the graph of $\vec{r}$ over the interval from $t = 0$ to $t = 7$.

$$\vec{r}\,'(t) = -2t\,\hat{\imath} + 3t^2\,\hat{\jmath} + \sqrt{5}\,t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 9t^4 + 5t^2}$$

$$= 3|t|\sqrt{1+t^2}$$

$$= 3t\sqrt{1+t^2}, \quad t \geq 0$$

$$s(t) = \int_0^t 3\tau\sqrt{1+\tau^2}\,d\tau$$

$u = 1+\tau^2$

$du = 2\tau\,d\tau$

$$s(t) = \int_{u=1}^{u=1+t^2} \frac{3}{2}u^{1/2}\,du = u^{3/2}\Big|_1^{1+t^2}$$

$$\boxed{s(t) = (1+t^2)^{3/2} - 1}$$

$$\boxed{s(7) = 50^{3/2} - 1 \approx 352.55}$$

2. (10 points) In 1988, East German Petra Felke set a women's world record by throwing a javelin 262.4 ft (over 80 m!). Felke launched the javelin from 6.5 ft above the ground at 40° angle with the horizontal. What was the javelin's initial speed? And when did it reach it's maximum height? (To receive full credit, you must write and use the vector-valued function $\vec{r}(t)$ that gives the position of the object at time t. Also ignore air resistance and use $g \approx 32\,\text{ft}\,\text{s}^{-2}$.)

$$\vec{r}(t) = V_0\cos 40^\circ\, t\,\hat{\imath} + \left(-16t^2 + V_0\sin 40^\circ\, t + 6.5\right)\hat{\jmath}$$

$$V_0\cos 40^\circ\, t = 262.4 \Rightarrow V_0 = \frac{262.4}{t\cos 40^\circ}$$

$$-16t^2 + V_0\sin 40^\circ\, t + 6.5 = 0$$

$$-16t^2 + 262.4\tan 40^\circ + 6.5 = 0$$

$$t^2 = \frac{262.4\tan 40^\circ + 6.5}{16}$$

$$t \approx 3.76397 \text{ sec.}$$

$$\Downarrow$$

$$V_0 = \frac{262.4}{t\cos 40^\circ} \approx \boxed{91.00\ \text{FT/SEC}}$$

MAX HEIGHT WHEN

$$-32t + V_0\sin 40^\circ = 0$$

$$t = \frac{V_0\sin 40^\circ}{32}$$

$$\approx \boxed{1.828\ \text{SEC}}$$

3. (10 points) The total resistance, R, of two resistors connected in parallel satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2},$$

where R_1 and R_2 are the resistances of the connected resistors. Determine the total differential dR. Then use differentials to approximate ΔR as (R_1, R_2) changes from $(10, 15)$ to $(10.4, 14.7)$.

$$R = \left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-1} \Rightarrow R_{R_1} = -1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-2}\left(-\frac{1}{R_1^2}\right)$$

$$\Rightarrow R_{R_2} = -1\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-2}\left(-\frac{1}{R_2^2}\right)$$

AT $(10,15)$, $R_{R_1} = 0.36$

$R_{R_2} = 0.16$

$$\Delta R \approx R_{R_1}\Delta R_1 + R_{R_2}\Delta R_2 = (0.36)(0.4) + (0.16)(-0.3)$$

$$= \boxed{0.096}$$

4. (10 points) Consider the surface described by the equation $\sqrt{\frac{z+x}{y-1}} = z^2$. (Advice: This problem will be easier if you start by doing some algebra to rewrite the equation.)

(a) Find an equation of the plane tangent to the surface at the point $(3, 5, 1)$.

$$\underbrace{z^4(y-1) - (z+x)}_{F(x,y,z)} = 0$$

$$\vec{\nabla} F(x,y,z) = -\hat{i} + z^4\hat{j} + \left(4z^3(y-1)-1\right)\hat{k}$$

$$\vec{n} = \vec{\nabla} F(3,5,1) = -\hat{i} + \hat{j} + 15\hat{k}$$

OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$, WHICH PASSES THROUGH $(3,5,1)$

TANGENT PLANE: $-x + y + 15z = 17$

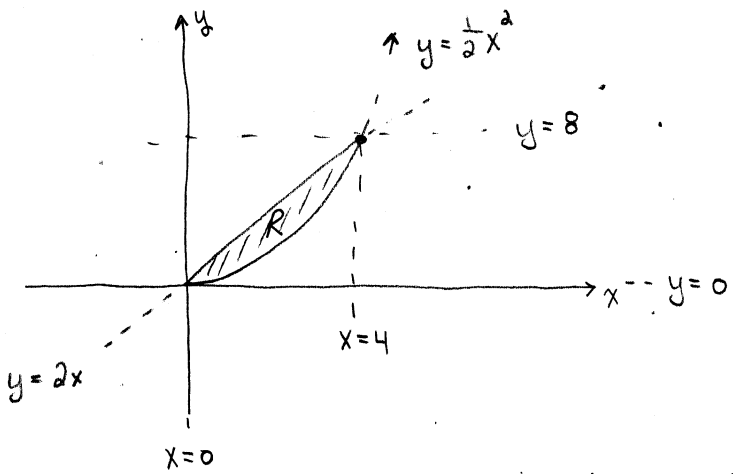
(b) Find symmetric equations for the line normal to the surface at $(3, 5, 1)$.

$x = -t + 3$
$y = t + 5$
$z = 15t + 1$

$$\Rightarrow \boxed{\frac{x-3}{-1} = y - 5 = \frac{z-1}{15}}$$

5. (10 points) The region R is the 1st quadrant region bounded by the graphs of $y = 2x$ and $y = \frac{1}{2}x^2$. Sketch the region of integration, write the double integral as an iterated integral with both orders of integration, and evaluate either one of your iterated integrals.

$$\iint_R (2x^2 + 8y)\, dA$$



$$\int_{x=0}^{x=4} \int_{y=\frac{1}{2}x^2}^{y=2x} (2x^2 + 8y)\, dy\, dx$$

$$= \int_{y=0}^{y=8} \int_{x=\frac{1}{2}y}^{x=\sqrt{2y}} (2x^2 + 8y)\, dx\, dy$$

$$2x = \frac{1}{2}x^2$$

$$\Downarrow$$

$$x^2 - 4x = 0$$

$$\Downarrow$$

$$x = 0,\ x = 4$$

$$\int_{x=0}^{x=4} \int_{y=\frac{1}{2}x^2}^{y=2x} (2x^2 + 8y)\, dy\, dx = \int_0^4 \left. 2x^2 y + 4y^2 \right|_{\frac{1}{2}x^2}^{2x} dx$$

$$= \int_0^4 (4x^3 + 16x^2 - x^4 - x^4)\, dx = \int_0^4 (4x^3 + 16x^2 - 2x^4)\, dx$$

$$= (4)^4 + \frac{16}{3}(4)^3 - \frac{2}{5}(4)^5 = \boxed{\frac{2816}{15}}$$

6. (10 points) Use Lagrange multipliers to find the minimum and maximum values of $F(x,y,z) = 2x + y - 2z$ on the sphere $\underbrace{x^2 + y^2 + z^2}_{G(x,y,z)} = 4$.

$$\vec{\nabla} F(x,y,z) = \lambda \vec{\nabla} G(x,y,z)$$

$$x^2 + y^2 + z^2 = 4$$

$$x = \pm\frac{4}{3}, \quad y = \pm\frac{2}{3}, \quad z = \mp\frac{4}{3}$$

$$2 = \lambda 2x \qquad\qquad x = \frac{1}{\lambda}$$

$$1 = \lambda 2y \quad \Rightarrow \quad y = \frac{1}{2\lambda}$$

$$-2 = \lambda 2z \qquad\qquad z = -\frac{1}{\lambda}$$

$$\boxed{\begin{array}{ll} F\left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right) = 6 & \text{ABS MAX} \\ F\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) = -6 & \text{ABS MIN} \end{array}}$$

$$\frac{1}{\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 4$$

$$\frac{9}{4\lambda^2} = 4 \Rightarrow \lambda^2 = \frac{9}{16} \Rightarrow \lambda = \pm\frac{3}{4}$$

7. (10 points) Evaluate $\int_C \frac{x}{1+y^2}\,ds$, where the curve C is made up of two line segments: the first from $\underbrace{(1,0) \text{ to } (3,1)}$, and the second from $\underbrace{(3,1) \text{ to } (5,1)}$.

$$\vec{PQ} = 2\hat{i} + \hat{j} \qquad\qquad \vec{PQ} = 2\hat{i}$$

$$x = 2t + 1 \quad 0 \le t \le 1 \qquad\qquad x = 2t + 3 \quad 0 \le t \le 1$$

$$y = t \qquad\qquad y = 1$$

$$\vec{r}(t) = (2t+1)\hat{i} + t\hat{j} \qquad\qquad \vec{r}(t) = (2t+3)\hat{i} + \hat{j}$$

$$\vec{r}\,'(t) = 2\hat{i} + \hat{j} \qquad\qquad \vec{r}\,'(t) = 2\hat{i}$$

$$\|\vec{r}\,'(t)\| = \sqrt{5} \qquad\qquad \|\vec{r}\,'(t)\| = 2$$

$$ds = \sqrt{5}\,dt \qquad\qquad ds = 2\,dt$$

1^{ST} PORTION ...

$$\sqrt{5}\int_0^1 \frac{2t+1}{1+t^2}\,dt = \sqrt{5}\ln|1+t^2| + \sqrt{5}\tan^{-1}t \Big|_0^1$$

$$= \sqrt{5}\ln 2 + \frac{\sqrt{5}\pi}{4}$$

2^{ND} PORTION ...

$$\cancel{2}\int_0^1 \frac{2t+3}{\cancel{2}}\,dt$$

$$= t^2 + 3t \Big|_0^1$$

$$= 4$$

$$\boxed{\int_C \frac{x}{1+y^2}\,ds = \sqrt{5}\ln 2 + \frac{\sqrt{5}\pi}{4} + 4}$$